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A unified Floquet theory for discrete, continuous, and hybrid periodic linear systems

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ABSTRACT

In this paper, we study periodic linear systems on periodic time scales which include not only discrete and continuous dynamical systems but also systems with a mixture of discrete and continuous parts (e.g. hybrid dynamical systems). We develop a comprehensive Floquet theory including Lyapunov transformations and their various stability preserving properties, a unified Floquet theorem which establishes a canonical Floquet decomposition on time scales in terms of the generalized exponential function, and use these results to study homogeneous as well as nonhomogeneous periodic problems. Furthermore, we explore the connection between Floquet multipliers and Floquet exponents via monodromy operators in this general setting and establish a spectral mapping theorem on time scales. Finally, we show this unified Floquet theory has the desirable property that stability characteristics of the original system can be determined via placement of an associated (but time varying) system's poles in the complex plane. We include several examples to show the utility of this theory.

1. Introduction

It is widely known that the stability characteristics of a nonautonomous p-periodic linear system of differential or difference equations can be characterized completely by a corresponding autonomous linear system of differential or difference equations by a periodic Lyapunov transformation of variables [8,20,27]. One application of Lyapunov transformations has been in generating different state

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variable descriptions of linear time invariant systems because different state variable descriptions correspond to different and perhaps more advantageous points of view in determining the system's output characteristics. This is useful in signals and systems applications for the simple fact that different descriptions of state variables allow usage of linear algebra to design and study the internal structure of a system. Having the ability to change the internal structure without changing the input–output behavior of the system is useful for identifying implementations of these systems that optimize some performance criteria that may not be directly related to input–output behavior, such as numerical effects of round-off error in a computer-based systems implementation. For example, using a transformation of variables on a discrete time non-diagonal 2×2 system, one can obtain a diagonal system matrix which separates the state update into two decoupled first-order difference equations, and, because of its simple structure, this form of the state variable description is very useful for analyzing the system's properties [14].

Without question, the study of periodic systems in general and Floquet theory in particular has been central to the differential equations theorist for some time. Researchers have explored these topics for ordinary differential equations [8,12,13,19,24,25,29,31], partial differential equations [7,9,13,21], differential-algebraic equations [11,22], and discrete dynamical systems [2,20,30]. Certainly [23] is a landmark paper in the area. Not surprisingly, Floquet theory has wide ranging effects, including extensions from time varying linear systems to time varying nonlinear systems of differential equations of the form $x' = f(t,x)$, where $f(t,x)$ is smooth and ω-periodic in t. The paper by Shi [28] ensures the global existence of solutions and proves that this system is topologically equivalent to an autonomous system $y' = g(y)$ via an ω-periodic transformation of variables. The theory has also been extended by Weikard [31] to nonautonomous linear systems of the form $\dot{z} = A(x)z$ where $A : \mathbb{C} \to \mathbb{C}^{n\times n}$ is an ω-periodic function in the complex variable x, whose solutions are meromorphic. With the assumption that $A(x)$ is bounded at the ends of the period strip, it is shown that there exists a fundamental solution of the form $P(x)e^{Jx}$ with a certain constant matrix J and function P which is rational in the variable $e^{2\pi ix/\omega}$.

In Teplinskiĭ and Teplinskiĭ [30], Lyapunov transformations and discrete Floquet theory are extended to countable systems in $\ell_\infty(\mathbb{N},\mathbb{R})$. It is proved that the countable time varying system can be represented by a countable time invariant system provided its finite-dimensional approximations can also be represented by time invariant systems.

Lyapunov transformations and Floquet theory have also been used to analyze the stability characteristics of quasilinear systems with periodically varying parameters. Pandiyan and Sinha [24] introduced a new technique for the investigation of these systems based on the fact that all quasilinear periodic systems can be replaced by similar systems whose linear parts are time invariant, via the well-known Lyapunov–Floquet transformation.

In the paper by Demir [11], the equivalent of Floquet theory is developed for periodically time varying systems of linear DAEs: $\frac{d}{dt}(C(t)x) + G(t)x = 0$ where the $n \times n$ matrices $C(\cdot)$ (not full rank in general) and $G(\cdot)$ are periodic. This result is developed for a direct application to oscillators which are ubiquitous in physical systems: gravitational, mechanical, biological, etc., and especially in electronic and optical ones. For example, in radio frequency communication systems, they are used for frequency translation of information signals and for channel selection. Oscillators are also present in digital electronic systems which require a time reference, i.e., a clock signal, in order to synchronize operations. All physical systems, and in particular electronic ones, are corrupted by undesired perturbations such as random thermal noise, substrate and supply noise, etc. Hence, signals generated by practical oscillators are not perfectly periodic. This performance limiting factor in electronic systems is also analyzed in [11] and a theory for nonlinear perturbation analysis of oscillators described by a system of DAEs is developed.

The intent of this paper is to extend the current results of continuous and discrete Floquet theory to the more general case of an arbitrary periodic *time scale* (a special case of a *measure chain* [15,16]), defined as any closed subset of $\mathbb{R}$. In particular, the main result shows that there exists a not necessarily constant $n \times n$ matrix $R(t)$ such that $e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0)$, where $\Phi_A(t, t_0)$ is the transition matrix for the p-periodic system $x^\Delta(t) = A(t)x(t)$, and that the transition matrix can be represented by the product of a p-periodic Lyapunov transformation matrix and a time scale matrix exponential, i.e. $\Phi_A(t, t_0) = L(t)e_R(t, t_0)$, which is known as the *Floquet decomposition* of the transition matrix $\Phi_A(t, t_0)$. Even though the matrix $R(t)$ is in general time varying, because of its construction, one can still analyze the stability of the original p-periodic system by the eigenvalues of $R(t)$, just as in the familiar (and special) cases of discrete and continuous Floquet theory. We make a note that with this unified Floquet theory, the matrix $R(t)$ does become a constant matrix R in the cases that the time scale is $\mathbb{R}$ or $\mathbb{Z}$, as one would expect.

There has been work on generalizing the Floquet decomposition to the time scales case in the very nice paper by Ahlbrandt and Ridenhour [2]. However, there are some important distinctions between their work and this one. First, Ahlbrandt and Ridenhour use a different definition of a periodic time scale. Furthermore—and very importantly—their Floquet decomposition theorem employs the usual exponential function whereas our approach is in terms of the generalized time scale exponential function. Finally, we go on to develop a complete Floquet theory including Lyapunov transformations and stability, Floquet multipliers, Floquet exponents, and apply this theory to questions of stability of periodic linear systems.

Additionally, this paper develops a solution to the previously open question of the existence of a solution matrix $R(t)$ to the matrix equation $e_R(t, \tau) = M$, where $R(t)$ and M are $n \times n$ matrices, and M is constant and nonsingular. Whereas the question of a generalized time scales logarithm remains open, the solution contained within this paper offers a step towards fulfilling this gap in time scales analysis.

This paper is organized as follows. In Section 2 the generalized Lyapunov transformation for time scales is developed (cf. [4]) and it is shown that the change of variables using the time scales version of this transformation preserves the stability properties of the system. Then in Section 3, the notion of a periodic time scale is presented and the main theorem, the unified and extended version of Floquet's theorem, is introduced for the homogeneous and nonhomogeneous cases of a periodic system on a periodic time scale. Three examples are given in Section 4 to show that the unified theory of Floquet is functional in the cases $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and more interestingly, when $\mathbb{T} = \mathbb{P}_{1,1}$. Section 5 introduces the unified theorems for Floquet multipliers, Floquet exponents, as well as a generalized spectral mapping theorem for time scales. In Section 6, the examples from Section 4 are revisited and the theorems introduced in Section 5 are illustrated. We end the paper with the Conclusions, where the main ideas are stated and the results of the paper are summarized. In Appendix A, necessary definitions and results are stated to keep the paper relatively self-contained. An excellent introduction to the subject of time scales analysis can be found in Bohner and Peterson's introductory texts [5,6].

2. The Lyapunov transformation and stability

We begin by analyzing the stability preserving property associated with a change of variables using a Lyapunov transformation on the regressive time varying linear dynamic system

$$x^\Delta(t) = A(t)x(t), \qquad x(t_0) = x_0. \tag{2.1}$$

Definition 2.1. A *Lyapunov transformation* is an invertible matrix $L(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ with the property that, for some positive $\eta, \rho \in \mathbb{R}$,

$$\|L(t)\| \leqslant \rho \quad \text{and} \quad |\det L(t)| \geqslant \eta \tag{2.2}$$

for all $t \in \mathbb{T}$.

The two following lemmas can be found in the classic text by Aitken [3].

Lemma 2.2. *Suppose that $A(t)$ is an $n \times n$ invertible matrix for all $t \in \mathbb{T}$. If there exists a constant $\alpha > 0$ such that $\|A^{-1}(t)\| \leqslant \alpha$ for each t, then there exists a constant β such that $|\det A(t)| \geqslant \beta$ for all $t \in \mathbb{T}$.*

Lemma 2.3. *Suppose that $A(t)$ is an $n \times n$ invertible matrix for all $t \in \mathbb{T}$. Then*

$$\left\|A^{-1}(t)\right\| \leqslant \frac{\|A(t)\|^{n-1}}{|\det A(t)|}$$

for all $t \in \mathbb{T}$.

A consequence of Lemmas 2.2 and 2.3 is that the inverse of a Lyapunov transformation is also bounded. An equivalent condition to (2.2) is that there exists a $\rho > 0$ such that

$$\left\|L(t)\right\| \leqslant \rho \quad \text{and} \quad \left\|L^{-1}(t)\right\| \leqslant \rho$$

for all $t \in \mathbb{T}$.

Definition 2.4. The time varying linear dynamic equation (2.1) is called *uniformly stable* if there exists a positive constant γ such that for any t_0, $x(t_0)$ the corresponding solution satisfies

$$\left\|x(t)\right\| \leqslant \gamma \left\|x(t_0)\right\|, \quad t \geqslant t_0.$$

Uniform stability can also be characterized using the following theorem.

Theorem 2.5. *The time varying linear dynamic equation* (2.1) *is uniformly stable if and only if there exists a $\gamma > 0$ such that the transition matrix Φ_A satisfies $\|\Phi_A(t, t_0)\| \leqslant \gamma$ for all $t \geqslant t_0$ with $t, t_0 \in \mathbb{T}$.*

Definition 2.6. The time varying linear dynamic equation (2.1) is called *uniformly exponentially stable* if there exists positive constants γ, λ with $-\lambda \in \mathcal{R}^+$ such that for any t_0, $x(t_0)$ the corresponding solution satisfies

$$\left\|x(t)\right\| \leqslant \left\|x(t_0)\right\| \gamma e_{-\lambda}(t, t_0), \quad t \geqslant t_0.$$

Uniform exponential stability can also be characterized using the following theorem.

Theorem 2.7. *The time varying linear dynamic equation* (2.1) *is uniformly exponentially stable if and only if there exists a $\lambda, \gamma > 0$ with $-\lambda \in \mathcal{R}^+$ such that the transition matrix Φ_A satisfies*

$$\left\|\Phi_A(t, t_0)\right\| \leqslant \gamma e_{-\lambda}(t, t_0)$$

for all $t \geqslant t_0$ with $t, t_0 \in \mathbb{T}$.

The last stability definition given uses a uniformity condition to conclude exponential stability.

Definition 2.8. The linear state equation (2.1) is defined to be uniformly asymptotically stable if it is uniformly stable and given any $\delta > 0$, there exists a $T > 0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies

$$\left\|x(t)\right\| \leqslant \delta \left\|x(t_0)\right\|, \quad t \geqslant t_0 + T. \tag{2.3}$$

It is noted that the time T that must pass before the norm of the solution satisfies (2.3) and the constant $\delta > 0$ is independent of the initial time t_0.

Theorem 2.9. *The linear state equation* (2.1) *is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

Proof. Suppose that the system (2.1) is uniformly exponentially stable. This implies that there exist constants $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ so that $\|\Phi_A(t,\tau)\| \leqslant \gamma e_{-\lambda}(t,\tau)$ for $t \geqslant \tau$. Clearly, this implies uniform stability. Now, given a $\delta > 0$, we choose a sufficiently large positive constant $T > 0$ such that $t_0 + T \in \mathbb{T}$ and $e_{-\lambda}(t_0+T, t_0) \leqslant \frac{\delta}{\gamma}$. Then for any t_0 and x_0, and $t \geqslant T + t_0$ with $t, T + t_0 \in \mathbb{T}$,

$$\begin{aligned}\|x(t)\| &= \|\Phi_A(t,t_0)x_0\| \\ &\leqslant \|\Phi_A(t,t_0)\|\|x_0\| \\ &\leqslant \gamma e_{-\lambda}(t,t_0)\|x_0\| \\ &\leqslant \gamma e_{-\lambda}(t_0+T,t_0)\|x_0\| \\ &\leqslant \delta\|x_0\|, \quad t \geqslant t_0 + T.\end{aligned}$$

Thus, (2.1) is uniformly asymptotically stable.

Now suppose the converse. By definition of uniform asymptotic stability, (2.1) is uniformly stable. Thus, there exists a constant $\gamma > 0$ so that

$$\|\Phi_A(t,\tau)\| \leqslant \gamma, \quad \text{for all } t \geqslant \tau. \tag{2.4}$$

Choosing $\delta = \frac{1}{2}$, let T be a positive constant so that $t_0 + T \in \mathbb{T}$ and (2.3) is satisfied. Given a t_0 and letting x_a be so that $\|x_a\| = 1$,

$$\|\Phi_A(t_0+T,t_0)x_a\| = \|\Phi_A(t_0+T,t_0)\|.$$

When $x_0 = x_a$, the solution $x(t)$ of (2.1) satisfies

$$\|x(t_0+T)\| = \|\Phi_A(t_0+T,t_0)x_a\| = \|\Phi_A(t_0+T,t_0)\|\|x_a\| \leqslant \frac{1}{2}\|x_a\|.$$

From this, we obtain

$$\|\Phi_A(t_0+T,t_0)\| \leqslant \frac{1}{2}. \tag{2.5}$$

It can be seen that for any t_0 there exists an x_a as claimed. Therefore, the above inequality (2.5) holds for any $t_0 \in \mathbb{T}$.

Using the bound from the inequalities (2.4) and (2.5), we have the following set of inequalities on intervals in the time scale of the form $[\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ with $k \in \mathbb{Z}_0^+$ and arbitrary τ:

$$\begin{aligned}\|\Phi_A(t,\tau)\| &\leqslant \gamma, \quad t \in [\tau, \tau+T)_{\mathbb{T}}, \\ \|\Phi_A(t,\tau)\| &= \|\Phi_A(t,\tau+T)\Phi_A(\tau+T,\tau)\| \\ &\leqslant \|\Phi_A(t,\tau+T)\|\|\Phi_A(\tau+T,\tau)\| \\ &\leqslant \frac{\gamma}{2}, \quad t \in [\tau+T, \tau+2T)_{\mathbb{T}}, \\ \|\Phi_A(t,\tau)\| &= \|\Phi_A(t,\tau+2T)\Phi_A(\tau+2T,\tau+T)\Phi_A(\tau+T,\tau)\| \\ &\leqslant \|\Phi_A(t,\tau+2T)\|\|\Phi_A(\tau+2T,\tau+T)\|\|\Phi_A(\tau+T,\tau)\| \\ &\leqslant \frac{\gamma}{2^2}, \quad t \in [\tau+2T, \tau+3T)_{\mathbb{T}}.\end{aligned}$$

In general, for any $\tau \in \mathbb{T}$,

$$\left\|\Phi_A(t,\tau)\right\| \leqslant \frac{\gamma}{2^k}, \quad t \in \left[\tau + kT, \tau + (k+1)T\right)_{\mathbb{T}}.$$

We now determine a decaying exponential bound by finding a positive constant λ (with $-\lambda \in \mathcal{R}^+$) that satisfies $\frac{1}{2} \leqslant e_{-\lambda}(\tau + T, \tau) \leqslant e_{-\lambda}(t,\tau)$, for $t \in [\tau, \tau + T)_{\mathbb{T}}$. Recall the fact for any positive constant β such that $-\beta \in \mathcal{R}^+$,

$$\int_{t_0}^{t} \lim_{s \searrow \mu_{\max}} \xi_s(-\beta)\Delta\tau \leqslant \int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \xi_s(-\beta)\Delta\tau.$$

Then we must define λ such that

$$\ln\left(\frac{1}{2}\right) = \int_{t_0}^{t_0+T} \lim_{s \searrow \mu_{\max}} \xi_s(-\lambda)\Delta\tau.$$

It is easy to verify that

$$\lambda = \lim_{s \searrow \mu_{\max}} \frac{1 - (\frac{1}{2})^{\frac{s}{T}}}{s}.$$

Since T has been established to satisfy (2.3) with $\delta = \frac{1}{2}$, for any $k \in \mathbb{N}_0$ and $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$, where $\tau + kT$ and $\tau + (k+1)T$ are merely the upper bounds on the interval, not necessarily elements in the time scale,

$$\frac{1}{2^k} \leqslant e_{-\lambda}(t,\tau),$$

for $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$.

Then for all $t, \tau \in \mathbb{T}$ with $t \geqslant \tau$, we obtain the correct value of λ for the decaying exponential bound

$$\left\|\Phi_A(t,\tau)\right\| \leqslant \gamma e_{-\lambda}(t,\tau).$$

Therefore, by Theorem 2.7 we have uniform exponential stability. □

Theorem 2.10. *Suppose $L(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$, with $L(t)$ invertible for all $t \in \mathbb{T}$ and $A(t)$ is from the dynamic linear system* (2.1)*. Then the transition matrix for the system*

$$Z^{\Delta}(t) = G(t)Z(t), \qquad Z(\tau) = I \tag{2.6}$$

where

$$G(t) = L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t) \tag{2.7}$$

is given by

$$\Phi_G(t,\tau) = L^{-1}(t)\Phi_A(t,\tau)L(\tau)$$

for any $t, \tau \in \mathbb{T}$.

Proof. First we see that by definition, $G(t) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. For any $\tau \in \mathbb{T}$, we define

$$X(t) = L^{-1}(t)\Phi_A(t,\tau)L(\tau). \tag{2.8}$$

It is obvious that for $t = \tau$, $X(\tau) = I$. Temporarily rearranging (2.8) and differentiating $L(t)X(t)$ with respect to t, we obtain [6, Theorem 5.3 (iv)]

$$L^{\Delta}(t)X(t) + L^{\sigma}(t)X^{\Delta}(t) = \Phi_A^{\Delta}(t,\tau)L(\tau) = A(t)\Phi_A(t,\tau)L(\tau),$$

and thus

$$\begin{aligned} L^{\sigma}(t)X^{\Delta}(t) &= A(t)\Phi_A(t,\tau)L(\tau) - L^{\Delta}(t)X(t) \\ &= A(t)\Phi_A(t,\tau)L(\tau) - L^{\Delta}(t)L^{-1}(t)\Phi_A(t,\tau)L(\tau) \\ &= \left[A(t) - L^{\Delta}(t)L^{-1}(t)\right]\Phi_A(t,\tau)L(\tau). \end{aligned}$$

Multiplying both sides by $L^{\sigma^{-1}}(t)$ and noting (2.7) and (2.8),

$$\begin{aligned} X^{\Delta}(t) &= \left[L^{\sigma^{-1}}(t)A(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)L^{-1}(t)\right]\Phi_A(t,\tau)L(\tau) \\ &= \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)\right]L^{-1}(t)\Phi_A(t,\tau)L(\tau) \\ &= G(t)X(t). \end{aligned}$$

This is valid for any $\tau \in \mathbb{T}$. Thus, the transition matrix of $X^{\Delta}(t) = G(t)X(t)$ is $\Phi_G(t,\tau) = L^{-1}(t)\Phi_A(t,\tau)L(\tau)$. Additionally, if the initial value specified in (2.6) was not the identity, i.e. $Z(t_0) = Z_0 \neq I$, then the solution is $X(t) = \Phi_G(t,\tau)Z_0$. □

2.1. Preservation of uniform stability

Theorem 2.11. *Suppose that $z(t) = L^{-1}(t)x(t)$ is a Lyapunov transformation. Then the system* (2.1) *is uniformly stable if and only if*

$$z^{\Delta}(t) = \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)\right]z(t), \qquad z(t_0) = z_0 \tag{2.9}$$

is uniformly stable.

Proof. Eqs. (2.1) and (2.9) are related by the change of variables $z(t) = L^{-1}(t)x(t)$. By Theorem 2.10, the relationship between the two transition matrices is

$$\Phi_G(t,t_0) = L^{-1}(t)\Phi_A(t,t_0)L(t_0).$$

Suppose that (2.1) is uniformly stable. Then there exists a $\gamma > 0$ such that $\|\Phi_A(t,t_0)\| \leqslant \gamma$ for all $t, t_0 \in \mathbb{T}$ with $t \geqslant t_0$. Then by Lemma 2.3 and Theorem 2.5,

$$\begin{aligned} \left\|\Phi_G(t,t_0)\right\| &= \left\|L^{-1}(t)\Phi_A(t,t_0)L(t_0)\right\| \\ &\leqslant \left\|L^{-1}(t)\right\|\left\|\Phi_A(t,t_0)\right\|\left\|L(t_0)\right\| \\ &\leqslant \frac{\gamma\rho^n}{\eta}, \end{aligned}$$

for all $t, t_0 \in \mathbb{T}$ with $t \geqslant t_0$. By Theorem 2.5, since $\|\Phi_G(t,t_0)\| \leqslant \frac{\gamma\rho^n}{\eta}$, the system (2.9) is uniformly stable. The converse is similar. □

2.2. Preservation of uniform exponential stability

Theorem 2.12. *Suppose that $z(t) = L^{-1}(t)x(t)$ is a Lyapunov transformation. Then the system* (2.1) *is uniformly exponentially stable if and only if*

$$z^{\Delta}(t) = \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)\right]z(t), \qquad z(t_0) = z_0 \tag{2.10}$$

is uniformly exponentially stable.

Proof. Eqs. (2.1) and (2.10) are related by the change of variables $z(t) = L^{-1}(t)x(t)$. By Theorem 2.10, the relationship between the two transition matrices is

$$\Phi_G(t,t_0) = L^{-1}(t)\Phi_A(t,t_0)L(t_0).$$

Suppose that (2.1) is uniformly exponentially stable. Then there exists a $\lambda, \gamma > 0$ with $-\lambda \in \mathcal{R}^+$ such that $\|\Phi_A(t,t_0)\| \leqslant \gamma e_{-\lambda}(t,t_0)$ for all $t \geqslant t_0$ with $t, t_0 \in \mathbb{T}$. Then by Lemma 2.3 and Theorem 2.7,

$$\begin{aligned}\left\|\Phi_G(t,t_0)\right\| &= \left\|L^{-1}(t)\Phi_A(t,t_0)L(t_0)\right\| \\ &\leqslant \left\|L^{-1}(t)\right\|\left\|\Phi_A(t,t_0)\right\|\left\|L(t_0)\right\| \\ &\leqslant \frac{\gamma\rho^n}{\eta}e_{-\lambda}(t,t_0),\end{aligned}$$

for all $t, t_0 \in \mathbb{T}$ with $t \geqslant t_0$.

By Theorem 2.7, since $\|\Phi_G(t,t_0)\| \leqslant \frac{\gamma\rho^n}{\eta}e_{-\lambda}(t,t_0)$, the system (2.10) is uniformly exponentially stable. The converse is similar. □

Corollary 2.13. *Suppose that $z(t) = L^{-1}(t)x(t)$ is a Lyapunov transformation. Then the system* (2.1) *is uniformly asymptotically stable if and only if*

$$z^{\Delta}(t) = \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)\right]z(t), \qquad z(t_0) = z_0$$

is uniformly asymptotically stable.

Proof. The proof follows from Theorem 2.9. □

3. A unified Floquet theory

We want to make a vital distinction that is necessary for comprehension of the notation that will be used in the remainder of the paper. When considering solutions to the linear dynamic system (2.1), just as in the familiar case of $\mathbb{T} = \mathbb{R}$, we may write the transition matrix differently, depending on properties of the system matrix.

In the most general case, the solution may always be expressed as $x(t) = \Phi_A(t,t_0)x_0$, where Φ_A is written as [10, Theorem 3.2]

$$\Phi_A(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)\Delta\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\Delta\tau_2\Delta\tau_1 + \cdots + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\cdots\int_{t_0}^{\tau_{i-1}} A(\tau_i)\Delta\tau_i\cdots\Delta\tau_1 + \cdots, \tag{3.1}$$

which generalizes the classical derivation using Picard iterates for a first-order linear differential equation to any time scale.

When the system matrix commutes with its integral, i.e.

$$A(t)\int_{s}^{t} A(\tau)\Delta\tau = \int_{s}^{t} A(\tau)\Delta\tau\, A(t), \quad \text{for all } s,t \in [t_0,\infty)_{\mathbb{T}},$$

we may write the solution to (2.1) as $x(t) = e_A(t,t_0)x_0$. This type of system is also known as a *Lappo-Danilevskiĭ system* by the Russian school; see [1]. In this case, the representation of the matrix exponential e_A is equivalent to the transition matrix Φ_A in (3.1). However, the property that separates the transition matrix from the matrix exponential (defined as the solution to (2.1)) is that

$$A(t)e_A(t,t_0) = e_A(t,t_0)A(t) \quad \text{but in general} \quad A(t)\Phi_A(t,t_0) \neq \Phi_A(t,t_0)A(t).$$

Note that if the system matrix is constant, we may again represent the solution with the matrix exponential, either by using the previous representation (3.1) or the infinite series [10, Theorem 4.1]

$$e_A(t,t_0) = \sum_{k=0}^{\infty} A^k h_k(t,t_0).$$

Just as on $\mathbb{R}$ and $h\mathbb{Z}$, one of the main properties that distinguishes Φ_A from e_A in the context of a solution to a linear dynamic system of the form (2.1), is that in general Φ_A does not commute with A. However, when the solution to (2.1) may be expressed as e_A, it is equivalent to the fact that e_A and A necessarily commute for all $t \in \mathbb{T}$.

We now state definitions that will be used throughout the paper.

Definition 3.1. Let $p \in (0,\infty)$. Then the time scale $\mathbb{T}$ is *p-periodic* if for all $t \in \mathbb{T}$:

(i) $t \in \mathbb{T}$ implies $t + p \in \mathbb{T}$,
(ii) $\mu(t) = \mu(t+p)$.

Definition 3.2. $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is *p-periodic* if $A(t) = A(t+p)$ for all $t \in \mathbb{T}$.

3.1. The homogeneous equation

Let $\mathbb{T}$ be a p-periodic time scale. Consider the regressive time varying linear dynamic initial value problem

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \tag{3.2}$$

where $A(t)$ is p-periodic for all $t \in \mathbb{T}$ and the time scale $\mathbb{T}$ is also p-periodic. Although it is not necessary that the period of $A(t)$ and the period of the time scale to be equal, we will assume so for simplicity. Furthermore, we will assume that $0 \leqslant \mu(t) \leqslant p$ and that the linear dynamic system and the time scale under discussion are p-periodic, unless references to more general systems are made.

The following theorem is crucial for the development of the unified Floquet decomposition.

Remark. Only the principal values of the matrix functions and the eigenvalues will be considered in the remainder of the paper.

Theorem 3.3 *(Construction of the R matrix). Given a nonsingular constant matrix M and constant $p > 0$, a solution of the time scale matrix exponential equation $e_R(t_0 + p, t_0) = M$ is given by $R : \mathbb{T} \to \mathbb{C}^{n\times n}$, where*

$$R(t) := \lim_{s \searrow \mu(t)} \frac{M^{\frac{s}{p}} - I}{s} \tag{3.3}$$

and the matrix power is defined as in Appendix A. If $\mathbb{T}$ has constant graininess on the interval $[t_0, t_0 + p]_{\mathbb{T}}$, then $R(t)$ is constant.

Proof. Observe that

$$R(t) \int_{t_0}^{t} R(\tau)\Delta\tau = \int_{t_0}^{t} R(\tau)\Delta\tau\, R(t),$$

for all $t, t_0 \in \mathbb{T}$. We can represent the transition matrix solution $\Phi_R(t, t_0)$ of the linear dynamic system

$$z^{\Delta}(t) = R(t)z(t), \qquad z(t_0) = z_0,$$

using the matrix exponential. In other words, due to the commutativity of $R(t)$,

$$\Phi_R(t, t_0) \equiv e_R(t, t_0).$$

It can be verified by direct calculation that $e_R^{\Delta}(t, t_0) = R(t)e_R(t, t_0) = e_R(t, t_0)R(t)$ for all $t \in \mathbb{T}$. Using (3.3), we obtain

$$e_R(t, t_0) = M^{\frac{t-t_0}{p}}.$$

To prove this claim, first note that $e_R(t_0, t_0) = M^0 = I$. Delta differentiating,

$$\begin{aligned} e_R^{\Delta}(t, t_0) &= \lim_{s \searrow \mu(t)} \frac{M^{\frac{t+s-t_0}{p}} - M^{\frac{t-t_0}{p}}}{s} \\ &= \left(\lim_{s \searrow \mu(t)} \frac{M^{\frac{s}{p}} - I}{s}\right) M^{\frac{t-t_0}{p}} \\ &= R(t)e_R(t, t_0). \end{aligned}$$

It follows that

$$e_R(t_0 + p, t_0) = M^{\frac{t_0+p-t_0}{p}} = M. \qquad \square$$

An interesting and useful property of the matrix R constructed in Theorem 3.3 is stated in the following corollary.

Corollary 3.4. *The matrices $R(t)$ and M have identical eigenvectors.*

Proof. Implementing Theorem A.8, for any of the n eigenpairs $\{\lambda_i, v_i\}$ of M,

$$Mv_i = \lambda_i v_i \implies \lim_{s\searrow\mu(t)} M^{\frac{s}{p}} v_i = \lim_{s\searrow\mu(t)} \lambda_i^{\frac{s}{p}} v_i \implies R(t)v_i = \lim_{s\searrow\mu(t)} \left(\frac{\lambda_i^{\frac{s}{p}} - 1}{s}\right) v_i.$$

Thus, the n eigenpairs of $R(t)$ are $\{\gamma_i(t), v_i\}_{i=1}^n$, where $\gamma_i(t) := \lim_{s\searrow\mu(t)} \frac{\lambda_i^{\frac{s}{p}}-1}{s}$. $\square$

Lemma 3.5. *Suppose that $\mathbb{T}$ is a p-periodic time scale and $P(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ is also p-periodic. Then the solution of the dynamic matrix initial value problem*

$$Z^{\Delta}(t) = P(t)Z(t), \qquad Z(t_0) = Z_0, \tag{3.4}$$

is unique up to a period p shift. That is, $\Phi_P(t, t_0) = \Phi_P(t + kp, t_0 + kp)$, for all $t \in \mathbb{T}$ and $k \in \mathbb{N}_0$.

Proof. By [6], the unique solution to (3.4) is $Z(t) = \Phi_P(t, t_0)Z_0$. Observe

$$\Phi_P^{\Delta}(t, t_0)Z_0 = P(t)\Phi_P(t, t_0)Z_0 \quad \text{and} \quad \Phi_P(t, t_0)|_{t=t_0} Z_0 = \Phi_P(t_0, t_0)Z_0 = Z_0.$$

Now we show that $\Phi_P(t, t_0) = \Phi_P(t + kp, t_0 + kp)$, for all $t \in \mathbb{T}$ and $k \in \mathbb{N}_0$. We do so by observing that $\Phi_P(t + kp, t_0 + kp)Z_0$ also solves the matrix initial value problem (3.4). We see that

$$\begin{aligned}
\Phi_P^{\Delta_{t+kp}}(t + kp, t_0 + kp)Z_0 &= P(t + kp)\Phi_P(t + kp, t_0 + kp) \\
&= P(t)\Phi_P(t + kp, t_0 + kp), \quad \text{and} \\
\Phi_P(t + kp, t_0 + kp)|_{t+kp=t_0+kp} &= \Phi_P(t + kp, t_0 + kp)|_{t=t_0} \\
&= \Phi_P(t_0 + kp, t_0 + kp)Z_0 \\
&= Z_0.
\end{aligned}$$

Since the solution to (3.4) is unique, we conclude

$$\Phi_P(t + kp, t_0 + kp) = \Phi_P(t, t_0), \quad \text{for all } t \in \mathbb{T} \text{ and } k \in \mathbb{N}_0. \qquad \square$$

The next theorem is the unified and extended version of the Floquet decomposition for p-periodic time varying linear dynamic systems.

Theorem 3.6 *(The unified Floquet decomposition). The transition matrix for a p-periodic $A(t)$ can be written in the form*

$$\Phi_A(t, \tau) = L(t)e_R(t, \tau)L^{-1}(\tau) \quad \text{for all } t, \tau \in \mathbb{T}, \tag{3.5}$$

where $R : \mathbb{T} \to \mathbb{C}^{n\times n}$ and $L(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{C}^{n\times n})$ are both p-periodic and invertible at each $t \in \mathbb{T}$. We refer to (3.5) *as the* Floquet decomposition for Φ_A.

Proof. We define the matrix R as in Theorem 3.3, with $M := \Phi_A(t_0 + p, t_0)$. Using this definition, R satisfies the equation

$$e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0).$$

Define the matrix $L(t)$ by

$$L(t) := \Phi_A(t, t_0) e_R^{-1}(t, t_0). \tag{3.6}$$

By definition, $L(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{C}^{n\times n})$ and is invertible at each $t \in \mathbb{T}$. Now

$$\Phi_A(t, t_0) = L(t) e_R(t, t_0), \tag{3.7}$$

yields

$$\Phi_A(t_0, t) = e_R^{-1}(t, t_0) L^{-1}(t) = e_R(t_0, t) L^{-1}(t),$$

which, together with (3.6), implies

$$\Phi_A(t, \tau) = L(t) e_R(t, \tau) L^{-1}(\tau),$$

for all $\tau, t \in \mathbb{T}$.

We conclude by showing that $L(t)$ is p-periodic. By (3.6) and Lemma 3.5,

$$\begin{aligned}
L(t+p) &= \Phi_A(t+p, t_0) e_R^{-1}(t+p, t_0) \\
&= \Phi_A(t+p, t_0+p) \Phi_A(t_0+p, t_0) e_R(t_0, t_0+p) e_R(t_0+p, t+p) \\
&= \Phi_A(t, t_0) \Phi_A(t_0+p, t_0) e_R^{-1}(t_0+p, t_0) e_R(t_0, t) \\
&= \Phi_A(t, t_0) e_R^{-1}(t, t_0) \\
&= L(t). \qquad \square
\end{aligned}$$

Theorem 3.7. *Let $\Phi_A(t, t_0) = L(t) e_R(t, t_0)$ be a Floquet decomposition for Φ_A. Then $x(t) = \Phi_A(t, t_0) x_0$ is a solution of the p-periodic system* (3.2) *if and only if $z(t) = L^{-1}(t) x(t)$ is a solution of the system*

$$z^{\Delta}(t) = R(t) z(t), \qquad z(t_0) = x_0. \tag{3.8}$$

Proof. Suppose $x(t)$ is a solution to (3.2). Then $x(t) = \Phi_A(t, t_0) x_0 = L(t) e_R(t, t_0) x_0$. Setting

$$z(t) := L^{-1}(t) x(t) = L^{-1}(t) L(t) e_R(t, t_0) x_0 = e_R(t, t_0) x_0,$$

it follows from the construction of $R(t)$ that $z(t)$ is a solution of (3.8).

Now suppose $z(t) = L^{-1}(t) x(t)$ is a solution of the system (3.8). Then $z(t) = e_R(t, t_0) x_0$. Set $x(t) := L(t) z(t)$. Then

$$x(t) = L(t) e_R(t, t_0) x_0 = \Phi_A(t, t_0) x_0,$$

and thus $x(t)$ is a solution of (3.2). □

Corollary 3.8. *The solutions of the system* (3.2) *are uniformly stable* (*respectively, uniformly exponentially stable, asymptotically stable*) *if and only if the solutions of the system* (3.8) *are uniformly stable* (*respectively, uniformly exponentially stable, asymptotically stable*).

Proof. The proof follows from the fact that the systems are related by a Lyapunov change of variables and implementation of the appropriate stability preservation theorem in Section 2. □

Theorem 3.9. *Given any $t_0 \in \mathbb{T}$, there exists an initial state $x(t_0) = x_0 \neq 0$ such that the solution of* (3.2) *is p-periodic if and only if at least one of the eigenvalues of $e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0)$ is* 1.

Proof. Suppose that given an initial time t_0 with $x(t_0) = x_0 \neq 0$, the solution $x(t)$ is p-periodic. By Theorem 3.6, there exists a Floquet decomposition of x given by

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)L^{-1}(t_0)x_0.$$

Furthermore,

$$x(t + p) = L(t + p)e_R(t + p, t_0)L^{-1}(t_0)x_0 = L(t)e_R(t + p, t_0)L^{-1}(t_0)x_0.$$

Since $x(t) = x(t + p)$ and $L(t) = L(t + p)$ for each $t \in \mathbb{T}$,

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t + p, t_0)L^{-1}(t_0)x_0,$$

which implies

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t + p, t_0 + p)e_R(t_0 + p, t_0)L^{-1}(t_0)x_0.$$

Since $e_R(t + p, t_0 + p) = e_R(t, t_0)$,

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t, t_0)e_R(t_0 + p, t_0)L^{-1}(t_0)x_0,$$

and thus

$$L^{-1}(t_0)x_0 = e_R(t_0 + p, t_0)L^{-1}(t_0)x_0.$$

Since $L^{-1}(t_0)x_0 \neq 0$, we see that $L^{-1}(t_0)x_0$ is an eigenvector of the matrix $e_R(t_0+p, t_0)$ corresponding to an eigenvalue of 1.

Now suppose 1 is an eigenvalue of $e_R(t_0 + p, t_0)$ with corresponding eigenvector z_0. Then z_0 is real-valued and nonzero. For any $t_0 \in \mathbb{T}$, $z(t) = e_R(t, t_0)z_0$ is p-periodic. Since 1 is an eigenvalue of $e_R(t_0 + p, t_0)$ with corresponding eigenvector z_0 and $e_R(t + p, t_0 + p) = e_R(t, t_0)$,

$$\begin{aligned} z(t + p) &= e_R(t + p, t_0)z_0 \\ &= e_R(t + p, t_0 + p)e_R(t_0 + p, t_0)z_0 \\ &= e_R(t + p, t_0 + p)z_0 \\ &= e_R(t, t_0)z_0 \\ &= z(t). \end{aligned}$$

Using the Floquet decomposition from Theorem 3.6 and setting $x_0 := L(t_0)z_0$, we obtain the nontrivial solution of (3.2). Then

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)L^{-1}(t_0)x_0 = L(t)e_R(t, t_0)z_0 = L(t)z(t),$$

which is p-periodic since $L(t)$ and $z(t)$ are p-periodic. □

3.2. The nonhomogeneous equation

We now consider the nonhomogeneous uniformly regressive time varying linear dynamic initial value problem

$$x^{\Delta}(t) = A(t)x(t) + f(t), \qquad x(t_0) = x_0, \tag{3.9}$$

where $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $f \in \mathrm{C_{prd}}(\mathbb{T}, \mathbb{R}^{n\times 1}) \cap \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times 1})$, and both are p-periodic for all $t \in \mathbb{T}$.

Lemma 3.10. *A solution $x(t)$ of* (3.9) *is p-periodic if and only if $x(t_0 + p) = x(t_0)$.*

Proof. Suppose that $x(t)$ is p-periodic. Then by definition of a periodic function, $x(t_0 + p) = x(t_0)$.

Now suppose that there exists a solution of (3.9) such that $x(t_0 + p) = x(t_0)$. Define $z(t) = x(t + p) - x(t)$. By assumption and construction of $z(t)$, we have $z(t_0) = 0$. Furthermore,

$$\begin{aligned} z^{\Delta}(t) &= \big[A(t+p)x(t+p) + f(t+p)\big] - \big[A(t)x(t) + f(t)\big] \\ &= A(t)\big[x(t+p) - x(t)\big] \\ &= A(t)z(t). \end{aligned}$$

By uniqueness of solutions, we see that $z(t) \equiv 0$ for all $t \in \mathbb{T}$. Thus, $x(t) = x(t + p)$ for all $t \in \mathbb{T}$. □

The next theorem uses Lemma 3.10 to develop criteria for the existence of p-periodic solutions for any p-periodic vector-valued function $f(t)$.

Theorem 3.11. *For all $t_0 \in \mathbb{T}$ and for all p-periodic $f(t)$, there exists an initial state $x(t_0) = x_0$ such that the solution of* (3.9) *is p-periodic if and only if there does not exist a nonzero $z(t_0) = z_0$ and $t_0 \in \mathbb{T}$ such that the homogeneous initial value problem*

$$z^{\Delta}(t) = A(t)z(t), \qquad z(t_0) = z_0, \tag{3.10}$$

(*where $A(t)$ is p-periodic*) *has a p-periodic solution.*

Proof. The solution of (3.9) is given by

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A\big(t, \sigma(\tau)\big) f(\tau)\Delta\tau.$$

From Lemma 3.10, $x(t)$ is p-periodic if and only if $x(t_0) = x(t_0 + p)$ which is equivalent to

$$\big[I - \Phi_A(t_0 + p, t_0)\big]x_0 = \int_{t_0}^{t_0+p} \Phi_A\big(t_0 + p, \sigma(\tau)\big) f(\tau)\Delta\tau. \tag{3.11}$$

By Theorem 3.9, we must show that this algebraic equation has a solution for any initial condition $x(t_0)$ and any p-periodic $f(t)$ if and only if $e_R(t_0 + p, t_0)$ has no eigenvalues equal to one.

First, suppose $e_R(t_1 + p, t_1) = \Phi_A(t_1 + p, t_1)$, for some $t_1 \in \mathbb{T}$, and suppose there are no eigenvalues equal to one. This is equivalent to

$$\det\big[I - \Phi_A(t_1 + p, t_1)\big] \neq 0.$$

Since Φ_A is p-periodic and invertible, we obtain the equivalent statement

$$\begin{aligned} 0 &\neq \det\big[\Phi_A(t_0+p,t_1+p)\big(I-\Phi_A(t_1+p,t_1)\big)\Phi_A(t_1,t_0)\big] \\ &= \det\big[\Phi_A(t_0+p,t_1+p)\Phi_A(t_1,t_0)-\Phi_A(t_0+p,t_0)\big]. \end{aligned} \tag{3.12}$$

Since $\Phi_A(t_0+p,t_1+p)=\Phi_A(t_0,t_1)$, (3.12) is equivalent to the invertibility of $[I-\Phi_A(t_0+p,t_0)]$, for any $t_0\in\mathbb{T}$. Thus, for any initial time $t_0\in\mathbb{T}$ and for any p-periodic $f(t)$, (3.11) has a solution and that solution is

$$x_0=\big[I-\Phi_A(t_0+p,t_0)\big]^{-1}\int_{t_0}^{t_0+p}\Phi_A\big(t_0+p,\sigma(\tau)\big)f(\tau)\Delta\tau.$$

Now suppose (3.11) has a solution for every t_0 and every p-periodic $f(t)$. Given an arbitrary $t_0\in\mathbb{T}$, corresponding to any f_0, we define a regressive p-periodic vector-valued function $f(t)\in \mathrm{C}_{\mathrm{prd}}(\mathbb{T},\mathbb{R}^{n\times 1})$ by

$$f(t)=\Phi_A\big(\sigma(t),t_0+p\big)f_0,\quad t\in[t_0,t_0+p)_{\mathbb{T}}, \tag{3.13}$$

extending this to the entire time scale $\mathbb{T}$ using the periodicity.

By construction of $f(t)$,

$$\int_{t_0}^{t_0+p}\Phi_A\big(t_0+p,\sigma(\tau)\big)f(\tau)\Delta\tau=\int_{t_0}^{t_0+p}f_0\Delta\tau=pf_0.$$

Thus, (3.11) becomes

$$\big[I-\Phi_A(t_0+p,t_0)\big]x_0=pf_0. \tag{3.14}$$

For any $f(t)$ that is constructed as in (3.13), and thus for any corresponding f_0, (3.14) has a solution for x_0 by assumption. Therefore,

$$\det\big[I-\Phi_A(t_0+p,t_0)\big]\neq 0.$$

Hence, $e_R(t_0+p,t_0)=\Phi_A(t_0+p,t_0)$ has no eigenvalue of equal to 1. By Theorem 3.9, (3.10) has no periodic solution. □

4. Examples

4.1. Discrete time example

Consider the time scale $\mathbb{T}=\mathbb{Z}$ and the regressive (on $\mathbb{Z}$) time varying matrix

$$A(t)=\begin{bmatrix} -1 & \frac{2+(-1)^t}{2} \\ \frac{2+(-1)^t}{2} & -1 \end{bmatrix},$$

which have periods of 1 and 2, respectively. The transition matrix for the homogeneous periodic linear system of difference equations

$$\Delta X(t) = \begin{bmatrix} -1 & \frac{2+(-1)^t}{2} \\ \frac{2+(-1)^t}{2} & -1 \end{bmatrix} X(t), \tag{4.1}$$

is (from straightforward calculation) given by

$$\Phi_A(t,0) = \frac{1}{2^{t+1}} \begin{bmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{bmatrix}.$$

As in Theorem 3.6, consider the equation

$$e_R(2,0) = \Phi_A(2,0) = \frac{1}{2^3} \begin{bmatrix} (\sqrt{3})^2 + (-\sqrt{3})^2 & (\sqrt{3})^3 + (-\sqrt{3})^3 \\ (\sqrt{3})^3 + (-\sqrt{3})^3 & (\sqrt{3})^2 + (-\sqrt{3})^2 \end{bmatrix}$$

which simplifies to

$$e_R(2,0) = (I+R)^2 = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}.$$

Then $R(t)$ in the Floquet decomposition is given by

$$R(t) := \lim_{s \searrow 1} \frac{1}{s}\big(\Phi_A(2,0)^{\frac{s}{2}} - I\big) = \begin{bmatrix} \frac{\sqrt{3}}{2} - 1 & 0 \\ 0 & \frac{\sqrt{3}}{2} - 1 \end{bmatrix}, \tag{4.2}$$

which is constant (as expected) since $\mu(t) \equiv 1$ on $\mathbb{Z}$. Furthermore, the 2-periodic matrix $L(t)$ is given by

$$\begin{aligned} L(t) &:= \Phi_A(t,0) e_R^{-1}(t,0) \\ &= \Phi_A(t,0)(I+R)^{-t} \\ &= \frac{1}{2^{t+1}} \begin{bmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{bmatrix} \begin{bmatrix} (\frac{\sqrt{3}}{2})^{-t} & 0 \\ 0 & (\frac{\sqrt{3}}{2})^{-t} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + (-1)^t & \sqrt{3} + (-1)^t(-\sqrt{3}) \\ \sqrt{3} + (-1)^t(-\sqrt{3}) & 1 + (-1)^t \end{bmatrix}. \end{aligned} \tag{4.3}$$

Using R and $L(t)$ above, it is straightforward to verify that indeed (3.7), and equivalently, (3.5) both hold here.

4.2. Continuous time example

Consider the time scale $\mathbb{T} = \mathbb{R}$ and the time varying matrix

$$A(t) = \begin{bmatrix} -1 & 0 \\ \sin(t) & 0 \end{bmatrix}$$

which has period 2π. The transition matrix for the homogeneous periodic linear system of differential equations

$$\dot{X}(t) = \begin{bmatrix} -1 & 0 \\ \sin t & 0 \end{bmatrix} X(t) \tag{4.4}$$

is given by

$$\Phi_A(t,0) = \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}(\cos t + \sin t)}{2} & 1 \end{bmatrix}.$$

As in Theorem 3.6, consider the equation

$$e_R(2\pi,0) = \Phi_A(2\pi,0) = \begin{bmatrix} e^{-2\pi} & 0 \\ \frac{1}{2} - \frac{e^{-2\pi}}{2} & 1 \end{bmatrix}.$$

Then $R(t)$ in the Floquet decomposition is given by

$$R(t) := \lim_{s \searrow 0} \frac{1}{s}\left(\Phi_A(2\pi,0)^{\frac{s}{2\pi}} - I\right) = \frac{1}{2\pi} \operatorname{Log} \Phi_A(2\pi,0) = \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \tag{4.5}$$

which is constant (as expected) since $\mu(t) \equiv 0$ on $\mathbb{R}$. Hence,

$$e_R(t,0) = e^{Rt} = \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}}{2} & 1 \end{bmatrix} \quad \text{and thus} \quad e^{-Rt} = \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2} - \frac{e^{t}}{2} & 1 \end{bmatrix}.$$

Furthermore, the 2π-periodic matrix $L(t)$ is given by

$$\begin{aligned} L(t) &= \Phi_A(t,0) e_R^{-1}(t,0) \\ &= \Phi_A(t,0) e^{-Rt} \\ &= \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}(\cos t + \sin t)}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2} - \frac{e^{t}}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos t + \sin t}{2} & 1 \end{bmatrix}. \end{aligned} \tag{4.6}$$

Using R and $L(t)$ above, it is straightforward to verify that indeed (3.7), and equivalently, (3.5) both hold here.

4.3. Hybrid example

This example highlights the scope and generality of Theorem 3.6 on domains with nonconstant graininess.

Consider the time scale $\mathbb{T} = \mathbb{P}_{1,1}$ which has a period of 2 and the regressive (on $\mathbb{P}_{1,1}$) time varying matrix

$$A(t) = \begin{bmatrix} g(t) & 1 \\ 0 & -3 \end{bmatrix}, \quad g(t) = -3 + \sin(2\pi t),$$

with period 1. The transition matrix for the homogeneous periodic linear system of dynamic equations

$$X^{\Delta}(t) = \begin{bmatrix} g(t) & 1 \\ 0 & -3 \end{bmatrix} X(t), \tag{4.7}$$

is given by

$$\Phi_A(t,0) = \begin{bmatrix} e_g(t,0) & C(t) \\ 0 & e_{-3}(t,0) \end{bmatrix}, \quad \text{where } C(t) := \int_0^t e_g\big(t,\sigma(\tau)\big) e_{-3}(\tau,0)\Delta\tau.$$

Then $R(t)$ in the Floquet decomposition is given by

$$\begin{aligned} R(t) &:= \lim_{s \searrow \mu(t)} \frac{1}{s}\big(\Phi_A(2,0)^{\frac{s}{2}} - I\big) \\ &= \lim_{s \searrow \mu(t)} \frac{1}{s}\left(\begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{s}{2}} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right), \end{aligned} \tag{4.8}$$

which is nonconstant (as expected). Hence,

$$e_R(t,0) = \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}},$$

and thus,

$$e_R^{-1}(t,0) = \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{-\frac{t}{2}}.$$

Furthermore, the 2-periodic matrix $L(t)$ is given by

$$\begin{aligned} L(t) &= \Phi_A(t,0) e_R^{-1}(t,0) \\ &= \begin{bmatrix} e_g(t,0) & C(t) \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{-\frac{t}{2}} \end{aligned} \tag{4.9}$$

which is obviously p-periodic on $\mathbb{P}_{1,1}$. Thus, the Floquet decomposition of the transition matrix is $\Phi_A(t,0) = L(t)e_R(t,0)$.

5. Floquet multipliers and Floquet exponents

Suppose that $\Phi_A(t,t_0)$ is the transition matrix and $\Phi(t)$ is the fundamental matrix at $t = \tau$ (i.e. $\Phi(\tau) = I$) for the system (3.2). Then we can write any fundamental matrix $\Psi(t)$ as

$$\Psi(t) = \Phi(t)\Psi(\tau) \quad \text{or} \quad \Psi(t) = \Phi_A(t,t_0)\Psi(t_0).$$

Definition 5.1. Let $x_0 \in \mathbb{R}^n$ be a nonzero vector and $\Psi(t)$ be any fundamental matrix for the system (3.2). The vector solution of the system with initial condition $x(t_0) = x_0$ is given by $\Phi_A(t,t_0)x_0$. The operator $M : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$M(x_0) := \Phi_A(t_0 + p, t_0)x_0 = \Psi(t_0 + p)\Psi^{-1}(t_0)x_0,$$

is called a *monodromy operator*. The eigenvalues of the monodromy operator are called the *Floquet* (or *characteristic*) *multipliers* of the system (3.2).

The following theorem establishes that characteristic multipliers are nonzero complex numbers intrinsic to the periodic system—they do not depend on the choice of the fundamental matrix. This result is analogous to the theorem dealing with the eigenvalues and invertibility of monodromy operators in [8], which can also be referenced for proof.

Theorem 5.2. *The following statements hold for the system* (3.2).

(1) *Every monodromy operator is invertible. In particular, every characteristic multiplier is nonzero.*
(2) *If M_1 and M_2 are monodromy operators, then they have the same eigenvalues. In particular, there are exactly n characteristic multipliers, counting multiplicities.*

With the Floquet normal form $\Phi_A(t,t_0) = \Psi_1(t)\Psi_1^{-1}(t_0) = L(t)e_R(t,t_0)L^{-1}(t_0)$ of the transition matrix for the system (3.2) on one hand, and the monodromy operator representation

$$M(x_0) = \Phi_A(t_0+p,t_0)x_0 = \Psi_1(t_0+p)\Psi_1^{-1}(t_0)x_0,$$

on the other, together we conclude

$$\Phi_A(t_0+p,t_0) = \Psi_1(t_0+p)\Psi_1^{-1}(t_0) = L(t_0)e_R(t_0+p,t_0)L^{-1}(t_0).$$

Thus, the Floquet (or characteristic) multipliers of the system are the eigenvalues of the matrix $e_R(t_0+p,t_0)$. The (possibly complex) scalar function $\gamma(t)$ is a *Floquet* (*or characteristic*) *exponent* of the p-periodic system (3.2) if λ is a Floquet multiplier and $e_\gamma(t_0+p,t_0) = \lambda$.

Theorem 5.3 will help us answer the question of whether or not the eigenvalues of the matrix $R(t)$ in the Floquet decomposition $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ are in fact Floquet exponents.

Theorem 5.3 *(Spectral mapping theorem for time scales). Suppose that $R(t)$ is an $n \times n$ matrix as in Theorem* 3.3, *with eigenvalues $\gamma_1(t),\dots,\gamma_n(t)$, repeated according to multiplicities. Then $\gamma_1^k(t),\dots,\gamma_n^k(t)$ are the eigenvalues of $R^k(t)$ and the eigenvalues of e_R are $e_{\gamma_1},\dots,e_{\gamma_n}$.*

Proof. By induction for the dimension n, we start by stating that the theorem is valid for 1×1 matrices. Suppose that it is true for all $(n-1) \times (n-1)$ matrices. For each fixed $t \in \mathbb{T}$, take $\gamma_1(t)$ and let $v \neq 0$ denote a corresponding eigenvector such that $R(t)v = \gamma_1(t)v$. Let $\mathbf{e}_1,\dots,\mathbf{e}_n$ denote the usual basis of $\mathbb{C}^n$. There exists a nonsingular matrix S such that $Sv = \mathbf{e}_1$. Thus we have $SR(t)S^{-1}\mathbf{e}_1 = \gamma_1(t)\mathbf{e}_1$, and the matrix $SR(t)S^{-1}$ has the block form

$$SR(t)S^{-1} = \begin{bmatrix} \gamma_1(t) & * \\ 0 & \tilde{R}(t) \end{bmatrix}.$$

The matrix $SR^k(t)S^{-1}$ has the same block form, only with block diagonal elements $\gamma_1^k(t)$ and $\tilde{R}^k(t)$. Clearly, the eigenvalues of this block matrix are $\gamma_1^k(t)$ together with the eigenvalues of $\tilde{R}^k(t)$. By induction, the eigenvalues of $\tilde{R}^k(t)$ are the kth powers of the eigenvalues of $\tilde{R}(t)$. This proves the first statement of the theorem.

Since we chose the matrix S so that $SR(t)S^{-1}$ is block diagonal, by construction, the matrix $e_{SRS^{-1}}$ is also block diagonal. We see that $e_{SRS^{-1}}$ has block diagonal form, with block diagonal elements e_{γ_1} and $e_{\tilde{R}}$. Using induction, it follows that the eigenvalues of $e_{\tilde{R}}$ are $e_{\gamma_2},\dots,e_{\gamma_n}$. Thus, the eigenvalues of $e_{SRS^{-1}} = Se_RS^{-1}$ are $e_{\gamma_1},\dots,e_{\gamma_n}$. □

We now know that the eigenvalues of the matrix $e_R(t_0+p,t_0)$ are the Floquet multipliers and the eigenvalues of $R(t)$ are Floquet exponents. However, in Theorem 5.5, we will see that although the Floquet exponents are the eigenvalues of the matrix $R(t)$, they are not unique. We first introduce the definition of a purely imaginary number on an arbitrary time scale.

Definition 5.4. Let $-\frac{\pi}{h} < \omega \leqslant \frac{\pi}{h}$. The *Hilger purely imaginary number* $\mathring{\imath}\omega$ is defined by $\mathring{\imath}\omega = \frac{e^{i\omega h}-1}{h}$. For $z \in \mathbb{C}_h$, we have that $\mathring{\imath}\operatorname{Im}_h(z) \in \mathbb{I}_h$. Also, when $h=0$, $\mathring{\imath}\omega = i\omega$.

Theorem 5.5 *(Nonuniqueness of Floquet exponents). Suppose $\gamma(t) \in \mathcal{R}$ is a (possibly complex) Floquet exponent, λ is the corresponding characteristic multiplier of the p-periodic system* (3.2) *such that $e_\gamma(t_0+p,t_0)=\lambda$, and $\mathbb{T}$ is a p-periodic time scale. Then $\gamma(t) \oplus \mathring{\imath}\frac{2\pi k}{p}$ is also a Floquet exponent for all $k \in \mathbb{Z}$.*

Proof. Observe that for any $k \in \mathbb{Z}$ and $t_0 \in \mathbb{T}$,

$$\begin{aligned}
e_{\gamma \oplus \mathring{\imath}\frac{2\pi k}{p}}(t_0+p,t_0) &= e_\gamma(t_0+p,t_0)e_{\mathring{\imath}\frac{2\pi k}{p}}(t_0+p,t_0) \\
&= e_\gamma(t_0+p,t_0)\exp\left(\int_{t_0}^{t_0+p} \frac{\operatorname{Log}(1+\mu(\tau)\mathring{\imath}\frac{2\pi k}{p})}{\mu(\tau)}\Delta\tau\right) \\
&= e_\gamma(t_0+p,t_0)\exp\left(\int_{t_0}^{t_0+p} \frac{\operatorname{Log}(1+\mu(\tau)\frac{e^{i2\pi k\mu(\tau)/p}-1}{\mu(\tau)})}{\mu(\tau)}\Delta\tau\right) \\
&= e_\gamma(t_0+p,t_0)\exp\left(\int_{t_0}^{t_0+p} \frac{\operatorname{Log}(e^{i2\pi k\mu(\tau)/p})}{\mu(\tau)}\Delta\tau\right) \\
&= e_\gamma(t_0+p,t_0)\exp\left(\int_{t_0}^{t_0+p} \frac{i2\pi k\mu(\tau)/p}{\mu(\tau)}\Delta\tau\right) \\
&= e_\gamma(t_0+p,t_0)\exp\left(\int_{t_0}^{t_0+p} \frac{i2\pi k}{p}\Delta\tau\right) \\
&= e_\gamma(t_0+p,t_0)e^{i2\pi k} \\
&= e_\gamma(t_0+p,t_0). \qquad \square
\end{aligned}$$

The next lemma will illustrate the periodic nature of a special exponential function which will be used in proving the nonuniqueness of Floquet exponents in Theorem 5.7

Lemma 5.6. *Let $\mathbb{T}$ be a p-periodic time scale and $k \in \mathbb{Z}$. Then the functions $e_{\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)$ and $e_{\ominus\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)$ are p-periodic.*

Proof. Let $t \in \mathbb{T}$. Then

$$\begin{aligned}
e_{\mathring{\imath}\frac{2\pi k}{p}}(t+p,t_0) &= \exp\left(\frac{i2\pi k(t+p-t_0)}{p}\right) \\
&= \exp\left(\frac{i2\pi k(t-t_0)}{p}\right)\exp\left(\frac{i2\pi kp}{p}\right)
\end{aligned}$$

$$= \exp\left(\frac{i2\pi k(t-t_0)}{p}\right)$$
$$= e_{\mathring{\imath}\frac{2\pi k}{p}}(t,t_0).$$

Therefore, $e_{\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)$ is a p-periodic function. The fact that $e_{\ominus\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)$ is p-periodic follows easily from the identity [6]

$$e_{\ominus\mathring{\imath}\frac{2\pi k}{p}}(t,t_0) = \frac{1}{e_{\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)}. \qquad \square$$

We now show that for any Floquet exponent, there exists a Floquet decomposition such that the Floquet exponent is an eigenvalue of the associated matrix $R(t)$.

Theorem 5.7 *(Special Floquet decompositions). If $\gamma(t)$ is a Floquet exponent for the system* (3.2) *and $\Phi_A(t,t_0)$ is the associated transition matrix, then there exists a Floquet decomposition of the form $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ where $\gamma(t)$ is an eigenvalue of $R(t)$.*

Proof. Consider the Floquet decomposition $\Phi_A(t,t_0) = \tilde{L}(t)e_{\tilde{R}}(t,t_0)$. By definition of the characteristic exponents, there is a characteristic multiplier λ such that $\lambda = e_\gamma(p+t_0,t_0)$, and, by Theorem 5.3, there is an eigenvalue $\tilde{\gamma}(t)$ of $\tilde{R}(t)$ such that $e_{\tilde{\gamma}}(p+t_0,t_0) = \lambda$. Also, by Theorem 5.5, there is some integer k such that $\tilde{\gamma}(t) = \gamma(t) \oplus \mathring{\imath}\frac{2\pi k}{p}$.

Set

$$R(t) := \tilde{R}(t) \ominus \mathring{\imath}\frac{2\pi k}{p} I,$$
$$L(t) := \tilde{L}(t)e_{\mathring{\imath}\frac{2\pi k}{p}I}(t,t_0).$$

By this definition it is implied that $\tilde{R}(t) = R(t) \oplus \mathring{\imath}\frac{2\pi k}{p}I$. Then $\gamma(t)$ is an eigenvalue of $R(t)$, $L(t)$ is a p-periodic function, and

$$L(t)e_R(t,t_0) = \tilde{L}(t)e_{\mathring{\imath}\frac{2\pi k}{p}I}(t,t_0)e_R(t,t_0) = \tilde{L}(t)e_{\mathring{\imath}\frac{2\pi k}{p}I\oplus R}(t,t_0) = \tilde{L}(t)e_{\tilde{R}}(t,t_0).$$

It follows that $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ is another Floquet decomposition where $\gamma(t)$ is an eigenvalue of $R(t)$. $\square$

The following theorem classifies the types of solutions that can arise with periodic systems.

Theorem 5.8. *If λ is a characteristic multiplier of the p-periodic system* (3.2) *and $e_\gamma(t_0+p,t_0) = \lambda$ for some $t_0 \in \mathbb{T}$, then there exists a* (*possibly complex*) *nontrivial solution of the form*

$$x(t) = e_\gamma(t,t_0)q(t)$$

where q is a p-periodic function. Moreover, for this solution $x(t+p) = \lambda x(t)$.

Proof. Let $\Phi_A(t,t_0)$ be the transition matrix for (3.2). By Theorem 5.7, there is a Floquet decomposition $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ such that $\gamma(t)$ is an eigenvalue of $R(t)$. There exists a vector $v \neq 0$ such that $R(t)v = \gamma(t)v$. It follows that $e_R(t,t_0)v = e_\gamma(t,t_0)v$, and therefore the solution $x(t) := \Phi_A(t,t_0)v$ can be represented in the form

$$x(t) = L(t)e_R(t,t_0)v = e_\gamma(t,t_0)L(t)v.$$

The solution required by the first part of the theorem is obtained by defining $q(t) := L(t)v$. The second part of the theorem follows from

$$\begin{aligned}
x(t+p) &= e_\gamma(t+p,t_0)q(t+p)\\
&= e_\gamma(t+p,t_0+p)e_\gamma(t_0+p,t_0)q(t)\\
&= e_\gamma(t_0+p,t_0)e_\gamma(t+p,t_0+p)q(t)\\
&= e_\gamma(t_0+p,t_0)e_\gamma(t,t_0)L(t)v\\
&= e_\gamma(t_0+p,t_0)x(t)\\
&= \lambda x(t). \qquad \square
\end{aligned}$$

Theorem 5.8 showed that if $\gamma(t)$ is a Floquet exponent of (3.2), then we can construct a nontrivial solution of the form $x(t) = e_\gamma(t,t_0)q(t)$, where $q(t)$ is p-periodic. In the next theorem it is shown that if two characteristic multipliers λ_1 and λ_2 of the system (3.2) are distinct, then as in Theorem 5.8, we can construct linearly independent solutions x_1 and x_2 of (3.2).

Theorem 5.9. *Suppose that λ_1, λ_2 are characteristic multipliers of the p-periodic system* (3.2) *and $\gamma_1(t), \gamma_2(t)$ are Floquet exponents such that $e_{\gamma_i}(t_0+p,t_0) = \lambda_i$, $i = 1,2$. If $\lambda_1 \neq \lambda_2$, then there exist p-periodic functions $q_1(t), q_2(t)$ such that*

$$x_1(t) = e_{\gamma_1}(t,t_0)q_1(t) \quad \textit{and} \quad x_2(t) = e_{\gamma_2}(t,t_0)q_2(t)$$

are linearly independent solutions of (3.2).

Proof. As in Theorem 5.7, let $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ be such that $\gamma_1(t)$ is an eigenvalue of $R(t)$ with corresponding (nonzero) eigenvector v_1. Since λ_2 is an eigenvalue of the monodromy matrix $\Phi_A(t_0+p,t_0)$, by Theorem 5.3 there is an eigenvalue $\gamma(t)$ of $R(t)$ such that $e_\gamma(t_0+p,t_0) = \lambda_2 = e_{\gamma_2}(t_0+p,t_0)$. Hence $\gamma_2(t) = \gamma(t) \oplus \mathring{\imath}\frac{2\pi k}{p}$ for some $k \in \mathbb{Z}$. Also, $\gamma(t) \neq \gamma_1(t)$ since $\lambda_1 \neq \lambda_2$. Thus, if v_2 is a nonzero eigenvector of $R(t)$ corresponding to the eigenvalue $\gamma(t)$, then the eigenvectors v_1 and v_2 are linearly independent.

As in the proof of Theorem 5.8, there are solutions of (3.2) of the form

$$x_1(t) = e_{\gamma_1}(t,t_0)L(t)v_1, \qquad x_2(t) = e_\gamma(t,t_0)L(t)v_2.$$

Because $x_1(t_0) = v_1$ and $x_2(t_0) = v_2$, these solutions are linearly independent. Finally, x_2 can be written as

$$x_2(t) = \left(e_{\gamma \oplus \mathring{\imath}\frac{2\pi k}{p}}(t,t_0)\right)\left(e_{\ominus \mathring{\imath}\frac{2\pi k}{p}}(t,t_0)L(t)v_2\right) = e_{\gamma_2}(t,t_0)\left(e_{\ominus \mathring{\imath}\frac{2\pi k}{p}}(t,t_0)L(t)v_2\right),$$

where $q_2(t) := e_{\ominus \mathring{\imath}\frac{2\pi k}{p}}(t,t_0)L(t)v_2$. $\square$

6. Examples revisited

We now revisit the examples from Section 4. We show for each of the three examples in Section 4 that the original $R(t)$ matrices and the corresponding Floquet exponents are not unique.

6.1. Discrete time example

In Section 4.1, the Floquet exponent we found for the system was $\frac{\sqrt{3}}{2}-1$. We show that $\gamma := -\frac{\sqrt{3}}{2}-1$ is also a Floquet exponent, but it is not an eigenvalue of the original matrix R in (4.2).

Set $\tilde{R} := R \ominus \mathring{\imath}2\pi Ik/p = R \ominus \mathring{\imath}\pi I$, with $k=1$ and $p=2$, as in Theorem 5.7. Then

$$\begin{aligned}\tilde{R}(t) &= \begin{bmatrix} \frac{\sqrt{3}}{2}-1 & 0 \\ 0 & \frac{\sqrt{3}}{2}-1 \end{bmatrix} \ominus \begin{bmatrix} \mathring{\imath}\pi & 0 \\ 0 & \mathring{\imath}\pi \end{bmatrix} \\ &= \begin{bmatrix} (\frac{\sqrt{3}}{2}-1)\ominus \mathring{\imath}\pi & 0 \\ 0 & (\frac{\sqrt{3}}{2}-1)\ominus \mathring{\imath}\pi \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\sqrt{3}}{2}-1 & 0 \\ 0 & -\frac{\sqrt{3}}{2}-1 \end{bmatrix}.\end{aligned}$$

Hence,

$$e_{\tilde{R}}(t,0) = (I+\tilde{R})^t = \begin{bmatrix} (-\frac{\sqrt{3}}{2})^t & 0 \\ 0 & (-\frac{\sqrt{3}}{2})^t \end{bmatrix},$$

and

$$e_{\mathring{\imath}\pi I}(t,0) = (-I)^t = \begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix}.$$

Next, using $L(t)$ from (4.3), set $\tilde{L}(t) := L(t)e_{\mathring{\imath}\pi I}(t,0)$ to obtain

$$\begin{aligned}\tilde{L}(t) &= \frac{1}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix} \\ &= \frac{(-1)^t}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}.\end{aligned}$$

Thus,

$$\tilde{L}(t)e_{\tilde{R}}(t,0) = L(t)e_{\mathring{\imath}\pi I}(t,0)e_{\tilde{R}}(t,0) = L(t)e_{\mathring{\imath}\pi I\oplus\tilde{R}}(t,0) = L(t)e_R(t,0),$$

and so

$$\begin{aligned}L(t)e_R(t,0) &= \frac{1}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} (\frac{\sqrt{3}}{2})^t & 0 \\ 0 & (\frac{\sqrt{3}}{2})^t \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix} \\ &\quad\times\begin{bmatrix} (-\frac{\sqrt{3}}{2})^t & 0 \\ 0 & (-\frac{\sqrt{3}}{2})^t \end{bmatrix} \\ &= \frac{(-1)^t}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} (-\frac{\sqrt{3}}{2})^t & 0 \\ 0 & (-\frac{\sqrt{3}}{2})^t \end{bmatrix} \\ &= \tilde{L}(t)e_{\tilde{R}}(t,0).\end{aligned}$$

Therefore, $\Phi_A(t,0) = \tilde{L}(t)e_{\tilde{R}}(t,0)$ is another (distinct) Floquet decomposition of the transition matrix for A. Moreover,

$$\gamma := (\sqrt{3}/2 - 1) \ominus \mathring{\imath}\pi = -\sqrt{3}/2 - 1,$$

is a Floquet exponent as well as an eigenvalue of R which corresponds to the Floquet multiplier $\lambda = \frac{3}{4}$; that is,

$$e_{(\sqrt{3}/2-1)\ominus\mathring{\imath}\pi}(2,0) = e_{-\sqrt{3}/2-1}(2,0) = 3/4.$$

6.2. Continuous time example

From Section 4.2, consider R from (4.5). Set $\tilde{R} := R \ominus \mathring{\imath}2\pi Ik/p = R - i\pi I$, with $k=1$ and $p = 2\pi$, as in Theorem 5.7. Then,

$$\tilde{R} = \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1-i & 0 \\ \frac{1}{2} & -i \end{bmatrix},$$

so that

$$e_{\tilde{R}}(t,0) = e^{\tilde{R}t} = \begin{bmatrix} e^{(-1-i)t} & 0 \\ \frac{e^{-it}}{2} - \frac{e^{(-1-i)t}}{2} & e^{-it} \end{bmatrix},$$

and

$$\Phi_A(2\pi,0) = e^{2\pi\tilde{R}} = \begin{bmatrix} e^{-2\pi} & 0 \\ \frac{1}{2} - \frac{e^{-2\pi}}{2} & 1 \end{bmatrix}.$$

Next, using $L(t)$ from (4.6), set $\tilde{L}(t) := L(t)e_{\mathring{\imath}I}(t,0) = L(t)e^{iIt}$ to obtain

$$\tilde{L}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos t + \sin t}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it} \end{bmatrix} = \begin{bmatrix} e^{it} & 0 \\ \frac{e^{it}}{2} - \frac{e^{it}(\cos t + \sin t)}{2} & e^{it} \end{bmatrix}.$$

Thus,

$$\tilde{L}(t)e_{\tilde{R}}(t,0) = L(t)e^{iIt}e^{\tilde{R}t} = L(t)e^{(iI+\tilde{R})t} = L(t)e^{Rt},$$

and so

$$\begin{aligned} L(t)e^{Rt} &= \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos t + \sin t}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos t + \sin t}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it} \end{bmatrix} \begin{bmatrix} e^{(-1-i)t} & 0 \\ \frac{e^{-it}}{2} - \frac{e^{(-1-i)t}}{2} & e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} e^{it} & 0 \\ \frac{e^{it}}{2} - \frac{e^{it}(\cos t + \sin t)}{2} & e^{it} \end{bmatrix} \begin{bmatrix} e^{(-1-i)t} & 0 \\ \frac{e^{-it}}{2} - \frac{e^{(-1-i)t}}{2} & e^{-it} \end{bmatrix} \\ &= \tilde{L}(t)e^{\tilde{R}t}. \end{aligned}$$

Therefore, $\Phi_A(t,0) = \tilde{L}(t)e_{\tilde{R}}(t,0)$ is another (distinct) Floquet decomposition of the transition matrix for A. Moreover, $\gamma_1 := -1 - i$, $\gamma_2 := -i$ are Floquet exponents as well as eigenvalues of R which correspond to the Floquet multipliers $\lambda_1 = e^{-2\pi}$, $\lambda_2 = 1$, respectively. That is, $e^{-2\pi - 2\pi i} = e^{-2\pi}$ and $e^{-2\pi i} = 1$.

6.3. Hybrid example

In Section 4.3, consider R from (4.8) which was given by

$$R(t) = \lim_{s \searrow \mu(t)} \frac{1}{s}\left(\begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{s}{2}} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right),$$

where $-2e^{-3} = e_{-3}(2,0)$ on $\mathbb{T}$. The eigenvalues of $R(t)$ are

$$\gamma_1(t) := \lim_{s \searrow \mu(t)} s^{-1}\big(\big(e_{-3}(2,0)\big)^{-\frac{s}{2}} - 1\big) \quad \text{and} \quad \gamma_2(t) := \lim_{s \searrow \mu(t)} s^{-1}\big(\big(e_{-3+\sin(2\pi t)}(2,0)\big)^{-\frac{s}{2}} - 1\big).$$

Set $\tilde{R} := R \ominus \mathring{\imath} 2\pi Ik/p = R \ominus \mathring{\imath}\pi I$, with $k = 1$ and $p = 2$, as in Theorem 5.7. Then,

$$\tilde{R}(t) = \lim_{s \searrow \mu(t)} \frac{1}{s}\left(\begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{s}{2}} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \ominus \begin{bmatrix} \mathring{\imath}\pi & 0 \\ 0 & \mathring{\imath}\pi \end{bmatrix},$$

and thus,

$$\begin{aligned} e_{\tilde{R}}(t,0) &= \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}} e_{\ominus \mathring{\imath}\pi I}(t,0) \\ &= \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}} e_{\mathring{\imath}\pi I}(0,t) \\ &= \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}} \begin{bmatrix} e^{-i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{bmatrix}. \end{aligned}$$

Next, using $L(t)$ from (4.9), set $\tilde{L}(t) := L(t)e_{\mathring{\imath}\pi I}(t,0)$ to obtain

$$\tilde{L}(t) = \begin{bmatrix} e_g(t,0) & C(t) \\ 0 & e_{-3}(t,0) \end{bmatrix} \begin{bmatrix} -2e^{-3} & C(2) \\ 0 & -2e^{-3} \end{bmatrix}^{-\frac{t}{2}} \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & e^{i\pi t} \end{bmatrix}.$$

Thus,

$$\tilde{L}(t)e_{\tilde{R}}(t,0) = L(t)e_{\mathring{\imath}\pi I}(t,0)e_{\tilde{R}}(t,0) = L(t)e_{\tilde{R}}(t,0)e_{\mathring{\imath}\pi I}(t,0) = L(t)e_{\tilde{R}\oplus\mathring{\imath}\pi I}(t,0) = L(t)e_R(t,0),$$

and so

$$\begin{aligned} &L(t)e_R(t,0) \\ &\quad = \left(\begin{bmatrix} e_g(t,0) & \int_0^t e_g(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix} \begin{bmatrix} -2e^{-3} & \int_0^2 e_g(2,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}^{-\frac{t}{2}}\right) \\ &\qquad \cdot \begin{bmatrix} -2e^{-3} & \int_0^2 e_g(2,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}} \end{aligned}$$

$$= \left(\begin{bmatrix} e_g(t,0) & \int_0^t e_g(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix} \begin{bmatrix} -2e^{-3} & \int_0^2 e_g(2,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}^{-\frac{t}{2}} \right)$$

$$\cdot \left(\begin{bmatrix} -2e^{-3} & \int_0^2 e_g(2,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}} \begin{bmatrix} e^{-i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{bmatrix} \right) \cdot \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & e^{i\pi t} \end{bmatrix}$$

$$= \left(\begin{bmatrix} e_g(t,0) & \int_0^t e_g(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix} \begin{bmatrix} -2e^{-3} & \int_0^2 e_g(2,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}^{-\frac{t}{2}} \right.$$

$$\left. \cdot \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & e^{i\pi t} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} -2e^{-3} & \int_0^2 e_g(2,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}^{\frac{t}{2}} \begin{bmatrix} e^{-i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{bmatrix} \right)$$

$$= \tilde{L}(t)e_{\tilde{R}}(t,0).$$

Therefore, $\Phi_A(t,0) = \tilde{L}(t)e_{\tilde{R}}(t,0)$ is another (distinct) Floquet decomposition of the transition matrix for A. Moreover, $\gamma(t) := -3 \ominus \mathring{\imath}\pi$ is another Floquet exponent as well as an eigenvalue of $R(t)$ which corresponds to the Floquet multipliers $\lambda_{1,2} = -2e^{-3}$. That is, $e_{-3\ominus\mathring{\imath}\pi}(2,0) = e_{-3}(2,0) = -2e^{-3}$.

7. Floquet theory, stability, and dynamic eigenpairs

Consider the p-periodic regressive system

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0. \tag{7.1}$$

Recall that the R matrix in the Floquet decomposition of Φ_A is given by

$$R(t) := \lim_{s \searrow \mu(t)} \frac{1}{s}\left(\Phi_A(t_0+p,t_0)^{\frac{s}{p}} - I\right), \tag{7.2}$$

and consider the uniformly regressive system associated with (7.1),

$$z^{\Delta}(t) = R(t)z(t), \qquad z(t_0) = x_0. \tag{7.3}$$

From Theorem 3.7, solutions to these two problems are related via $z(t) = L^{-1}(t)x(t)$ where $L(t)$ is the Lyapunov transformation from (3.6).

Given a constant $n \times n$ matrix M, let C be the nonsingular matrix that transforms M into its Jordan canonical form,

$$J := C^{-1}MC = \operatorname{diag}\left[J_{m_1}(\lambda_1), \ldots, J_{m_k}(\lambda_k)\right],$$

where $k \leqslant n$, $\sum_{i=1}^{k} m_i = n$, λ_i are the eigenvalues of M (where some of them may be equal), and $J_m(\lambda)$ is an $m \times m$ Jordan block,

$$J_m(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}. \tag{7.4}$$

We introduce a definition from [26] which gives a bound on the eigenvalues of a system. Then we state a lemma which proves that the system (7.3) associated with (7.1) via a Floquet decomposition is necessarily uniformly regressive.

Definition 7.1. (See [26].) The scalar function $\gamma : \mathbb{T} \to \mathbb{C}$ is *uniformly regressive* if there exists a constant $\delta > 0$ such that $0 < \delta^{-1} \leqslant |1 + \mu(t)\gamma(t)|$, for all $t \in \mathbb{T}^\kappa$.

Lemma 7.2. *Each eigenvalue of the matrix $R(t)$ in* (7.3) *is uniformly regressive.*

Proof. We show by direct substitution that the eigenvalues of $R(t)$ are uniformly regressive. Using Corollary 3.4, let $\gamma_i(t) := \lim_{s \searrow \mu(t)} \frac{\lambda_i^{\frac{s}{p}} - 1}{s}$ be any of the $k \leqslant n$ distinct eigenvalues of $R(t)$. Recall that in this p-periodic setting $0 \leqslant \mu(t) \leqslant p$. Suppose that $|\lambda_i| \geqslant 1$. Then,

$$\bigl|1 + \mu(t)\gamma_i(t)\bigr| = \lim_{s \searrow \mu(t)} \left|1 + s\frac{\lambda_i^{\frac{s}{p}} - 1}{s}\right| = \lim_{s \searrow \mu(t)} \bigl|\lambda_i^{\frac{s}{p}}\bigr| \geqslant 1.$$

On the other hand, suppose $0 \leqslant |\lambda_i| < 1$. Then,

$$\bigl|1 + \mu(t)\gamma_i(t)\bigr| = \lim_{s \searrow \mu(t)} \left|1 + s\frac{\lambda_i^{\frac{s}{p}} - 1}{s}\right| = \lim_{s \searrow \mu(t)} \bigl|\lambda_i^{\frac{s}{p}}\bigr| \geqslant |\lambda_i|.$$

Thus $\delta^{-1} := \min\{1, |\lambda_1|, \ldots, |\lambda_k|\}$ will suffice as the bound for uniform regressivity. □

Motivated by the work of Wu [32] on $\mathbb{R}$, we introduce the following definition for the time scales setting.

Definition 7.3. A nonzero, delta differentiable vector $w(t)$ is said to be a *dynamic eigenvector* of $M(t)$ associated with the *dynamic eigenvalue* $\xi(t)$ if the pair satisfies the *dynamic eigenvalue problem*

$$w^\Delta(t) = M(t)w(t) - \xi(t)w^\sigma(t), \quad t \in \mathbb{T}^\kappa. \tag{7.5}$$

We call $\{\xi(t), w(t)\}$ a *dynamic eigenpair*.

The nonzero, delta differentiable vector

$$m_i(t) := e_{\xi_i}(t, t_0)w_i(t), \tag{7.6}$$

is the *mode vector* of $M(t)$ associated with the dynamic eigenpair $\{\xi_i(t), w_i(t)\}$.

The following lemma proves that given any regressive linear dynamic system (not necessarily periodic), there always exists a set of n dynamic eigenpairs, with linearly independent dynamic eigenvectors.

Lemma 7.4. *Given the $n \times n$ regressive matrix M, there always exists a set of n dynamic eigenpairs with linearly independent dynamic eigenvectors. Each of the eigenpairs satisfies the vector dynamic eigenvalue problem* (7.5) *associated with M. Furthermore, when the n vectors form the columns of $W(t)$, then $W(t)$ satisfies the equivalent matrix dynamic eigenvalue problem*

$$W^\Delta(t) = M(t)W(t) - W^\sigma(t)\Xi(t), \quad \text{where } \Xi(t) := \operatorname{diag}\bigl[\xi_1(t), \ldots, \xi_n(t)\bigr]. \tag{7.7}$$

Proof. Let $\{\xi_i(t)\}_{i=1}^n$ be a set of (not necessarily distinct) regressive functions. Then the $n \times n$ nonsingular matrix $W(t)$ defined by

$$W(t) := \Phi_M(t, t_0) e_{\ominus \Xi}(t, t_0),$$

has as its columns the associated n linearly independent dynamic eigenvectors $\{w_i(t)\}_{i=1}^n$. By direct substitution into (7.7), the proof is complete. □

Next, we show that the stability of a linear dynamic system can be completely determined by the mode vectors m_i, $1 \leqslant i \leqslant n$, as constructed in (7.6).

Theorem 7.5 *(Stability via modal vectors). Solutions to the uniformly regressive* (*but not necessarily periodic*) *time varying linear dynamic system* (2.1) *are*:

(1) *stable if and only if there exists a $\gamma > 0$ such that every mode vector $m_i(t)$ of $A(t)$ satisfies $\|m_i(t)\| \leqslant \gamma < \infty$, $t > t_0$, for all $1 \leqslant i \leqslant n$,*
(2) *asymptotically stable if and only if, in addition to* (1), *$\|m_i(t)\| \to 0$, $t > t_0$, for all $1 \leqslant i \leqslant n$,*
(3) *exponentially stable if and only if there exists $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ such that $\|m_i(t)\| \leqslant \gamma e_\lambda(t, t_0)$, $t > t_0$, for all $1 \leqslant i \leqslant n$.*

Proof. Let $\{\xi_i(t), w_i(t)\}_{i=1}^n$ be a set of n dynamic eigenpairs with linearly independent dynamic eigenvectors associated with the system matrix in (2.1). The transition matrix can be represented by

$$\Phi_A(t, t_0) = W(t) e_\Xi(t, t_0) W^{-1}(t_0) \tag{7.8}$$

where $W(t) := [w_1(t), w_2(t), \dots, w_n(t)]$ and $\Xi(t) := \operatorname{diag}[\xi_1(t), \dots, \xi_n(t)]$.

Denoting the matrix $W^{-1}(t_0)$ as

$$W^{-1}(t_0) := \begin{bmatrix} v_1^T(t_0) \\ v_2^T(t_0) \\ \vdots \\ v_n^T(t_0) \end{bmatrix},$$

we now have the reciprocal basis of each $w_i(t_0)$ given by the row vectors $v_i^T(t_0)$.

Because $\Xi(t)$ is a diagonal matrix, (7.8) can be rewritten as

$$\Phi_A(t, t_0) = \sum_{i=1}^n e_{\xi_i}(t, t_0) W(t) F_i W^{-1}(t_0), \tag{7.9}$$

where $F_i := \delta_{ij}$ is $n \times n$. Observing that $v_i^T(t) w_j(t) = \delta_{ij}$ for all $t \in \mathbb{T}$, we may rewrite F_i as

$$F_i = W^{-1}(t)\big[0, \dots, 0, w_i(t), 0, \dots, 0\big]. \tag{7.10}$$

Substituting (7.10) into (7.9) yields

$$\Phi_A(t, t_0) = \sum_{i=1}^n e_{\xi_i}(t, t_0) w_i(t) v_i^T(t_0) = \sum_{i=1}^n m_i(t) v_i(t_0). \tag{7.11}$$

The proof concludes easily from (7.11). □

We now set the stage for the main result of the section. In the next theorem, we will show that given the system matrix $R(t)$ from (7.3), we can choose a set of n linearly independent dynamic eigenvectors that have a growth rate that is bounded by a finite sum of generalized polynomials.

Theorem 7.6. *Given the set of traditional eigenvalues $\{\gamma_i(t)\}_{i=1}^n$ from the matrix $R(t)$ in (7.2), let $\{w_i(t)\}_{i=1}^n$ denote the corresponding linearly independent dynamic eigenvectors as defined by Lemma 7.4. Then $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ is a set of dynamic eigenpairs of $R(t)$ with the property that each $w_i(t)$ is bounded by at most a finite sum of constant multiples of generalized polynomials. In other words, there exists positive constants $D_i > 0$ such that*

$$\|w_i(t)\| \leqslant D_i \sum_{k=0}^{m_i-1} h_k(t, t_0), \tag{7.12}$$

where m_i is the dimension of the Jordan block which contains the ith eigenvalue, for all $1 \leqslant i \leqslant n$.

Proof. The fact that $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ is a set of dynamic eigenpairs of $R(t)$ follows immediately from Lemma 7.4. We now show each dynamic eigenvector w_i satisfies the bound in (7.12). For an appropriately chosen nonsingular $n \times n$ constant matrix C, we define the Jordan form of the constant matrix $\Phi_A(t_0 + p, t_0)$ by

$$\begin{aligned} J &:= C^{-1}\Phi_A(t_0+p, t_0)C \\ &= \begin{bmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{m_d}(\lambda_d) \end{bmatrix}_{n\times n}, \end{aligned} \tag{7.13}$$

where $d \leqslant n$, $\sum_{i=1}^d m_i = n$, λ_i are the eigenvalues of $\Phi_A(t_0 + p, t_0)$ (where some of them may be equal), and $J_m(\lambda)$ is an $m \times m$ Jordan block as defined in (7.4).

Using the matrix C from (7.13), define

$$\begin{aligned} K(t) &:= C^{-1}R(t)C \\ &= C^{-1} \lim_{s\searrow\mu(t)} \frac{1}{s}\left(\Phi_A(t_0+p, t_0)^{\frac{s}{p}} - I\right)C \\ &= \lim_{s\searrow\mu(t)} \frac{1}{s}\left(C^{-1}\Phi_A(t_0+p, t_0)^{\frac{s}{p}}C - I\right) \\ &= \lim_{s\searrow\mu(t)} \frac{1}{s}\left(J^{\frac{s}{p}} - I\right), \end{aligned} \tag{7.14}$$

where the last equality is via Theorem A.6. Note that $K(t)$ has the block diagonal form

$$K(t) = \begin{bmatrix} K_1(t) & & & \\ & K_2(t) & & \\ & & \ddots & \\ & & & K_d(t) \end{bmatrix}_{n\times n}, \tag{7.15}$$

where each $K_i(t)$ from (7.15) has the form

$$K_i(t) := \lim_{s\searrow\mu(t)} \begin{bmatrix} \frac{\lambda_i^{\frac{s}{p}}-1}{s} & \frac{\lambda_i^{\frac{s}{p}-1}}{1!p} & \frac{(\frac{s}{p}-1)\lambda_i^{\frac{s}{p}-2}}{2!p} & \cdots & \frac{(\frac{s}{p}-1)\cdots(\frac{s}{p}-(n-2))\lambda_i^{\frac{s}{p}-(n-1)}}{(n-1)!p} \\ & \frac{\lambda_i^{\frac{s}{p}}-1}{s} & \frac{\lambda_i^{\frac{s}{p}-1}}{1!p} & \cdots & \frac{(\frac{s}{p}-1)\cdots(\frac{s}{p}-(n-3))\lambda_i^{\frac{s}{p}-(n-2)}}{(n-2)!p} \\ & & \frac{\lambda_i^{\frac{s}{p}}-1}{s} & \ddots & \vdots \\ & & & \ddots & \frac{\lambda_i^{\frac{s}{p}-1}}{1!p} \\ & & & & \frac{\lambda_i^{\frac{s}{p}}-1}{s} \end{bmatrix}_{m_i\times m_i}.$$

By hypothesis, the dynamic eigenvalues of $R(t)$ have been defined as its traditional eigenvalues $\gamma_i(t) = \lim_{s\searrow\mu(t)} \frac{\lambda_i^{\frac{s}{p}}-1}{s}$, for $1 \leqslant i \leqslant n$. Note that since $R(t)$ and $K(t)$ are similar, they have the same traditional eigenvalues, with corresponding multiplicities. Furthermore, choosing the n dynamic eigenvalues of $K(t)$ to be its n traditional eigenvalues, we claim the corresponding dynamic eigenvectors $\{u_i(t)\}_{i=1}^n$ are defined to be $u_i(t) := C^{-1}w_i(t)$.

To prove this, observe that since $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ is a set of dynamic eigenpairs for $R(t)$, by definition

$$w_i^\Delta(t) = R(t)w_i(t) - \gamma_i(t)w_i^\sigma(t),$$

for all $1 \leqslant i \leqslant n$. We can now show that $\{\gamma_i(t), u_i(t)\}_{i=1}^n$ is a set of dynamic eigenpairs for $K(t)$:

$$\begin{aligned} u_i^\Delta(t) &= C^{-1}w_i^\Delta(t) \\ &= C^{-1}R(t)w_i(t) - C^{-1}\gamma_i(t)w_i^\sigma(t) \\ &= K(t)C^{-1}w_i(t) - \gamma_i(t)C^{-1}w_i^\sigma(t) \\ &= K(t)u_i(t) - \gamma_i(t)u_i^\sigma(t), \end{aligned} \tag{7.16}$$

for all $1 \leqslant i \leqslant n$.

We now show that each of the $u_i(t)$ is bounded by a finite sum of constant multiples of generalized polynomials.

Since each of the n dynamic eigenpairs $\{\gamma_i(t), u_i(t)\}_{i=1}^n$ satisfy the dynamic eigenvalue problem (7.16), we can now find the structure of each of the dynamic eigenvectors. Choose the ith block of $K(t)$, with dimension $m_i \times m_i$. We must then solve the $m_i \times m_i$ linear dynamic system

$$\nu^\Delta(t) = \tilde{K}_i(t)\nu(t) = \begin{bmatrix} 0 & \frac{1}{\lambda_i p} & \frac{(\mu(t)-p)}{2\lambda_i^2 p^2} & \cdots & \frac{(\mu(t)-p)\cdots(\mu(t)-p(n-2))}{(n-1)!\lambda_i^{n-1}p^{n-1}} \\ & 0 & \frac{1}{\lambda_i p} & \cdots & \frac{(\mu(t)-p)\cdots(\mu(t)-p(n-3))}{(n-2)!\lambda_i^{n-2}p^{n-2}} \\ & & & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & \frac{1}{\lambda_i p} \\ & & & & 0 \end{bmatrix} \nu(t), \tag{7.17}$$

where $\tilde{K}_i(t) := K_i(t) \ominus \gamma_i(t)I$. Since there are m_i linearly independent solutions to (7.17), we will denote each of the $m_i \times 1$ solutions by $\nu_{i,j}(t)$, where i corresponds to the ith block matrix $K_i(t)$ of $K(t)$ and $j = 1, \dots, m_i$. For $1 \leqslant i \leqslant d$, define $l_i := \sum_{s=0}^{i-1} m_s$, with $m_0 := 0$. The form for an arbitrary $n \times 1$ column vector u_{l_i+j}, with $1 \leqslant j \leqslant m_i$, is

$$u_{l_i+j}^T(t) = \big[\underbrace{0, \dots, 0, \dots, 0}_{m_1+\dots+m_{i-1}}, \underbrace{\nu_{i,j}^T(t)}_{m_i}, \underbrace{0, \dots, 0, \dots, 0}_{m_{i+1}+\dots+m_d}\big]_{1\times n}. \tag{7.18}$$

By combining all n vector solutions from (7.16), the solution to the equivalent $n \times n$ matrix dynamic equation

$$U^{\Delta}(t) = K(t)U(t) - U^{\sigma}(t)\Gamma(t),$$

where $\Gamma(t) := \operatorname{diag}[\gamma_1(t), \dots, \gamma_n(t)]$, will have the form

$$U(t) := [u_1, \dots, u_{m_1}, \dots, u_{(\sum_{k=1}^{i-1} m_k)}, \dots, u_{(\sum_{k=1}^{i} m_k)}, \dots, u_{(\sum_{k=1}^{d} m_k)-1}, u_n]$$
$$= \begin{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m_1} \\ & v_{1,1} & \ddots & v_{1,m_1-1} \\ & & \ddots & \vdots \\ & & & v_{1,1} \end{bmatrix}_{m_1\times m_1} & & \\ & \ddots & \\ & & \begin{bmatrix} v_{d,1} & v_{d,2} & \cdots & v_{d,m_d} \\ & v_{d,1} & \ddots & v_{d,m_d-1} \\ & & \ddots & \vdots \\ & & & v_{d,1} \end{bmatrix}_{m_d\times m_d} \end{bmatrix}_{n\times n}.$$

For brevity, we focus on the m_i elements contained in the $m_i \times 1$ solution vector ν with the understanding that its transpose will be embedded into a $1 \times n$ vector of the form (7.18), in the appropriate place, so that it only acts on the corresponding block matrix $K_i(t)$ of $K(t)$ in (7.16). There are m_i linearly independent solutions of (7.17) and they have the form

$$\begin{aligned}
\nu_{i,1}(t) &:= \big[v_{i,m_i}(t), 0, \dots, 0\big]_{m_i\times 1}^T, \\
\nu_{i,2}(t) &:= \big[v_{i,m_i-1}(t), v_{i,m_i}(t), 0, \dots, 0\big]_{m_i\times 1}^T, \\
&\ \vdots \\
\nu_{i,m_i-1}(t) &:= \big[v_{i,2}(t), v_{i,3}(t), \dots, v_{i,m_i-1}(t), v_{i,m_i}(t), 0\big]_{m_i\times 1}^T, \\
\nu_{i,m_i}(t) &:= \big[v_{i,1}(t), v_{i,2}(t), v_{i,3}(t), \dots, v_{i,m_i-1}(t), v_{i,m_i}(t)\big]_{m_i\times 1}^T,
\end{aligned}$$

where the scalar functions $v_{i,j}$ are constructed by back solving based on the form of $\tilde{K}_i$ in (7.17). The associated dynamic equations are

$$\begin{aligned}
v_{i,m_i}^{\Delta}(t) &= 0, \\
v_{i,m_i-1}^{\Delta}(t) &= \frac{1}{\lambda_i p} v_{i,m_i}(t),
\end{aligned}$$

$$v_{i,m_i-2}^{\Delta}(t) = \frac{\mu(t)-p}{2\lambda_i^2 p^2} v_{i,m_i}(t) + \frac{1}{\lambda_i p} v_{i,m_i-1}(t),$$

$$v_{i,m_i-3}^{\Delta}(t) = \frac{(\mu(t)-p)(\mu(t)-2p)}{3!\lambda_i^3 p^3} v_{i,m_i}(t) + \frac{\mu(t)-p}{2\lambda_i^2 p^2} v_{i,m_i-1}(t) + \frac{1}{\lambda_i p} v_{i,m_i-2}(t),$$

$$\vdots$$

$$\begin{aligned} v_{i,2}^{\Delta}(t) &= \frac{(\mu(t)-p)(\mu(t)-2p)\cdots(\mu(t)-(m_i-3)p)}{(m_i-2)!\lambda_i^{m_i-2} p^{m_i-2}} v_{i,m_i}(t) \\ &\quad + \frac{(\mu(t)-p)(\mu(t)-2p)\cdots(\mu(t)-(m_i-4)p)}{(m_i-3)!\lambda_i^{m_i-3} p^{m_i-3}} v_{i,m_i-1}(t) + \cdots \\ &\quad + \frac{\mu(t)-p}{2\lambda_i^2 p^2} v_4(t) + \frac{1}{\lambda_i p} v_{i,3}(t), \\ v_{i,1}^{\Delta}(t) &= \frac{(\mu(t)-p)(\mu(t)-2p)\cdots(\mu(t)-(m_i-2)p)}{(m_i-1)!\lambda_i^{m_i-1} p^{m_i-1}} v_{i,m_i}(t) \\ &\quad + \frac{(\mu(t)-p)(\mu(t)-2p)\cdots(\mu(t)-(m_i-3)p)}{(m_i-2)!\lambda_i^{m_i-2} p^{m_i-2}} v_{i,m_i-1}(t) + \cdots \\ &\quad + \frac{\mu(t)-p}{2\lambda_i^2 p^2} v_{i,3}(t) + \frac{1}{\lambda_i p} v_{i,2}(t). \end{aligned} \tag{7.19}$$

The solutions to (7.19) are

$$v_{i,m_i}(t) = 1,$$

$$v_{i,m_i-1}(t) = \int_{t_0}^{t} \frac{1}{\lambda_i p} v_{i,m_i}(\tau)\Delta\tau,$$

$$v_{i,m_i-2}(t) = \int_{t_0}^{t} \frac{\mu(\tau)-p}{2\lambda_i^2 p^2} v_{i,m_i}(\tau)\Delta\tau + \int_{t_0}^{t} \frac{1}{\lambda_i p} v_{i,m_i-1}(\tau)\Delta\tau,$$

$$\begin{aligned} v_{i,m_i-3}(t) &= \int_{t_0}^{t} \frac{(\mu(\tau)-p)(\mu(\tau)-2p)}{3!\lambda_i^3 p^3} v_{i,m_i}(\tau)\Delta\tau + \int_{t_0}^{t} \frac{\mu(\tau)-p}{2\lambda_i^2 p^2} v_{i,m_i-1}(\tau)\Delta\tau \\ &\quad + \int_{t_0}^{t} \frac{1}{\lambda_i p} v_{i,m_i-2}(\tau)\Delta\tau, \end{aligned}$$

$$\vdots$$

$$v_{i,2}(t) = \int_{t_0}^{t} \frac{(\mu(\tau)-p)(\mu(\tau)-2p)\cdots(\mu(\tau)-(m_i-3)p)}{(m_i-2)!\lambda_i^{m_i-2} p^{m_i-2}} v_{i,m_i}(\tau)\Delta\tau$$

$$+\int_{t_0}^{t}\frac{(\mu(\tau)-p)(\mu(\tau)-2p)\cdots(\mu(\tau)-(m_i-4)p)}{(m_i-3)!\lambda_i^{m_i-3}p^{m_i-3}}v_{i,m_i-1}(\tau)\Delta\tau$$

$$+\cdots+\int_{t_0}^{t}\frac{\mu(\tau)-p}{2\lambda_i^2p^2}v_{i,4}(\tau)\Delta\tau+\int_{t_0}^{t}\frac{1}{\lambda_i p}v_{i,3}(\tau)\Delta\tau,$$

$$v_{i,1}(t)=\int_{t_0}^{t}\frac{(\mu(t)-p)(\mu(t)-2p)\cdots(\mu(t)-(m_i-2)p)}{(m_i-1)!\lambda_i^{m_i-1}p^{m_i-1}}v_{i,m_i}(t)\Delta\tau$$

$$+\int_{t_0}^{t}\frac{(\mu(\tau)-p)(\mu(\tau)-2p)\cdots(\mu(\tau)-(m_i-3)p)}{(m_i-2)!\lambda_i^{m_i-2}p^{m_i-2}}v_{i,m_i-1}(\tau)\Delta\tau$$

$$+\cdots+\int_{t_0}^{t}\frac{\mu(\tau)-p}{2\lambda_i^2p^2}v_{i,3}(t)\Delta\tau+\int_{t_0}^{t}\frac{1}{\lambda_i p}v_{i,2}\Delta\tau.$$

We can explicitly bound each $v_{i,j}$ which yields an explicit bound on each u_{ℓ_i+j}, for all $1\leqslant j\leqslant m_i$ and for all ℓ_i, with $1\leqslant i\leqslant d$. Recall $\mu(t)\leqslant\mu_{\max}\leqslant p$ for all $t\in\mathbb{T}$. There exists constants $B_{i,j}$, $i=1,\dots,d$, and $j=1,\dots,m_i$, such that

$$\left|v_{m_i}(t)\right|=1\leqslant B_{i,m_i}h_0(t,t_0)=B_{i,m_i},$$

$$\left|v_{m_i-1}(t)\right|\leqslant\int_{t_0}^{t}\frac{1}{\lambda_i p}v_{m_i}(\tau)\Delta\tau=\frac{h_1(t,t_0)}{\lambda_i p}\leqslant B_{i,m_i-1}h_1(t,t_0),$$

$$\begin{aligned}\left|v_{m_i-2}(t)\right|&\leqslant\int_{t_0}^{t}\left|\frac{\mu(\tau)-p}{2\lambda_i^2p^2}v_{m_i}(\tau)\right|\Delta\tau+\int_{t_0}^{t}\left|\frac{1}{\lambda_i p}v_{m_i-1}(\tau)\right|\Delta\tau\\&\leqslant\int_{t_0}^{t}\frac{p}{2\lambda_i^2p^2}\Delta\tau+\int_{t_0}^{t}\frac{h_1(\tau,t_0)}{\lambda_i^2p^2}\Delta\tau\\&\leqslant\frac{h_1(t,t_0)}{2\lambda_i^2p}+\frac{h_2(t,t_0)}{\lambda_i^2p^2}\\&\leqslant B_{i,m_i-2}\sum_{j=1}^{2}h_j(t,t_0),\end{aligned}$$

$$\begin{aligned}\left|v_{m_i-3}(t)\right|\leqslant&\int_{t_0}^{t}\left|\frac{(\mu(\tau)-p)(\mu(\tau)-2p)}{3!\lambda_i^3p^3}v_{m_i}(\tau)\right|\Delta\tau+\int_{t_0}^{t}\left|\frac{\mu(\tau)-p}{2\lambda_i^2p^2}v_{m_i-1}(\tau)\right|\Delta\tau\\&+\int_{t_0}^{t}\left|\frac{1}{\lambda_i p}v_{m_i-2}(\tau)\right|\Delta\tau\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{t_0}^{t} \left| \frac{2p^2}{3!\lambda_i^3 p^3} \right| \Delta\tau + \int_{t_0}^{t} \left| \frac{p}{2\lambda_i^2 p^2} \frac{h_1(\tau,t_0)}{\lambda_i p} \right| \Delta\tau + \int_{t_0}^{t} \left| \frac{1}{\lambda_i p} \left(\frac{h_1(\tau,t_0)}{2\lambda_i^2 p} + \frac{h_2(\tau,t_0)}{\lambda_i^2 p^2} \right) \right| \Delta\tau \\
&\leqslant B_{i,m_i-3} \sum_{j=1}^{3} h_j(t,t_0),
\end{aligned}$$

$$\begin{aligned}
\left| v_{m_i-4}(t) \right| &\leqslant \int_{t_0}^{t} \left| \frac{(\mu(\tau)-p)(\mu(\tau)-2p)(\mu(\tau)-3p)}{4!\lambda_i^4 p^4} v_{m_i}(\tau) \right| \Delta\tau \\
&\quad + \int_{t_0}^{t} \left| \frac{(\mu(\tau)-p)(\mu(\tau)-2p)}{3!\lambda_i^3 p^3} v_{m_i-1}(\tau) \right| \Delta\tau \\
&\quad + \int_{t_0}^{t} \left| \frac{\mu(\tau)-p}{2\lambda_i^2 p^2} v_{m_i-2}(\tau) \right| \Delta\tau + \int_{t_0}^{t} \left| \frac{1}{\lambda_i p} v_{m_i-3}(\tau) \right| \Delta\tau \\
&\leqslant \int_{t_0}^{t} \left| \frac{3!p^3}{4!\lambda_i^4 p^4} \right| \Delta\tau + \int_{t_0}^{t} \left| \frac{2p^2}{3!\lambda_i^3 p^3} \frac{h_1(\tau,t_0)}{\lambda_i p} \right| \Delta\tau + \int_{t_0}^{t} \left| \frac{p}{2\lambda_i^2 p^2} \left(\frac{h_1(\tau,t_0)}{2\lambda_i^2 p} + \frac{h_2(\tau,t_0)}{\lambda_i^2 p^2} \right) \right| \Delta\tau \\
&\quad + \int_{t_0}^{t} \left| \frac{1}{\lambda_i p} \left(\frac{h_1(\tau,t_0)}{3\lambda_i^3 p} + 2\frac{h_2(\tau,t_0)}{2\lambda_i^3 p^2} + \frac{h_3(\tau,t_0)}{\lambda_i^3 p^3} \right) \right| \Delta\tau \\
&\leqslant B_{i,m_i-4} \sum_{j=1}^{4} h_j(t,t_0), \\
&\ \ \vdots \\
|v_2| &\leqslant B_{i,2} \sum_{j=1}^{m_i-2} h_j(t,t_0), \\
|v_1| &\leqslant B_{i,1} \sum_{j=1}^{m_i-1} h_j(t,t_0).
\end{aligned}$$

For each $1 \leqslant i \leqslant d$, set $\beta_i := \max_{j=1,\ldots,m_i}\{B_{i,j}\}$. Then, for $1 \leqslant i \leqslant d$ and $j = 1,\ldots,m_i$,

$$\left\| u_{l_i+j}(t) \right\| \leqslant \beta_i \sum_{k=0}^{m_i-1} h_k(t,t_0).$$

Finally, recall that $w_i = Cu_i$ using C from (7.14) and define $D_i := \|C\|\beta_i$, for all $1 \leqslant i \leqslant n$. Then we have the bound

$$\left\| w_i(t) \right\| = \left\| Cu_i(t) \right\| \leqslant \|C\|\beta_i \sum_{k=0}^{m_i-1} h_k(t,t_0) = D_i \sum_{k=0}^{m_i-1} h_k(t,t_0),$$

which proves the second claim in the theorem. □

To complete the setup for the main theorem, we state a necessary result from [26] which characterizes the growth rates of generalized polynomials.

Lemma 7.7. *Let $\mathbb{T}$ be a time scale which is unbounded above but with bounded graininess. If $\lambda \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{C})$ satisfies*

$$\exists T \in \mathbb{T} \quad \textit{such that} \quad 0 < -\inf_{t \in [T,\infty)_{\mathbb{T}}} \mathrm{Re}_\mu\big(\lambda(t)\big),$$

then

$$\lim_{t \to \infty} h_k(t, t_0) e_\lambda(t, \tau) = 0, \quad \textit{for } \tau \in \mathbb{T},\ k \in \mathbb{N}_0.$$

The following definition will be used to obtain the main result of this section.

Definition 7.8. Let $\mathbb{C}_\mu := \{z \in \mathbb{C}\colon\ z \neq -\frac{1}{\mu(t)}\}$. Given an element $t \in \mathbb{T}^\kappa$ with $\mu(t) > 0$, we define the *Hilger circle* as

$$\mathcal{H}_t := \big\{z \in \mathbb{C}_\mu\colon\ \mathrm{Re}_\mu(z) < 0\big\}.$$

Likewise, when $\mu(t) = 0$, we define the Hilger circle as

$$\mathcal{H}_t := \{z \in \mathbb{C}\colon\ \mathrm{Re}\, z < 0\}.$$

We are now in position to prove the main stability result. The result will show that given a p-periodic time varying linear dynamic system (7.1), the traditional eigenvalues of the associated time varying linear dynamic system (7.3) (via the Floquet decomposition of Φ_A) completely determine the stability characteristics of the original system.

Theorem 7.9 *(Floquet stability theorem). Given the p-periodic system* (7.3) *with eigenvalues $\{\gamma_i(t)\}_{i=1}^n$, we have the following properties of the solutions to the original p-periodic system* (7.1):

(1) *if $\mathrm{Re}_\mu(\gamma_i(t)) < 0$ for all $i = 1, \ldots, n$, then the system* (7.1) *is exponentially stable*;
(2) *if $\mathrm{Re}_\mu(\gamma_i(t)) \leqslant 0$ for all $i = 1, \ldots, n$, and if, for each characteristic exponent with $\mathrm{Re}_\mu(\gamma_i(t)) = 0$, the algebraic multiplicity equals the geometry multiplicity, then the system* (7.1) *is stable*; *otherwise the system* (7.1) *is unstable, growing at rates of generalized polynomials of t*;
(3) *if $\mathrm{Re}_\mu(\gamma_i(t)) > 0$ for some $i = 1, \ldots, n$, then the system* (7.1) *is unstable.*

Proof. Let $\Phi_A(t, t_0)$ be the transition matrix for the system (7.1). Define the matrix $R(t)$ as in (7.2), and note that by Theorem 3.6, (7.3) is the associated system through the Floquet decomposition of Φ_A. Given the traditional eigenvalues $\{\gamma_i(t)\}_{i=1}^n$ of $R(t)$, we define a set of dynamic eigenpairs of $R(t)$ by $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ as in Lemma 7.4. By Theorem 7.6, the dynamic eigenvectors $w_i(t)$ are bounded by a finite sum of generalized polynomials as in (7.12). Employing Theorem 7.5, we can write the transition matrix of (7.3) as

$$e_R(t, t_0) = \sum_{i=1}^n m_i(t) v_i^T(t_0),$$

for all $1 \leqslant i \leqslant n$, where the vectors $m_i(t)$ and $v_i^T(t_0)$ are defined as in Theorem 7.5.

Case (1): Applying Lemma 7.7, for each $1 \leqslant i \leqslant n$,

$$\begin{aligned}\lim_{t\to\infty}\|m_i(t)\| &\leqslant \lim_{t\to\infty} D_i \sum_{k=0}^{d_i-1} h_k(t,t_0)\left|e_{\gamma_i}(t,t_0)\right| \\ &\leqslant \lim_{t\to\infty} C_\varepsilon e_{\mathrm{Re}_\mu[\gamma_i]\oplus\varepsilon}(t,t_0) \\ &\leqslant \lim_{t\to\infty} C_\varepsilon e_{-\lambda}(t,t_0) = 0,\end{aligned}$$

where d_i is the dimension of the Jordan block which contains the ith eigenvalue of $R(t)$, D_i is defined as in Theorem 7.6, $\varepsilon > 0$ is chosen so that $\gamma_i(t) \oplus \varepsilon \in \mathcal{H}_t$, for all $t \in \mathbb{T}$,

$$C_\varepsilon := \max_{t\geqslant t_0} D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) e_{\ominus\varepsilon}(t,t_0),$$

and $\lambda > 0$ is chosen such that $\mathrm{Re}_\mu[\gamma_i(t)] \oplus \varepsilon < -\lambda < 0$, for all $i = 1,\dots,n$ and $t \in \mathbb{T}$. By Theorem 7.5 (3), the system (7.3) is exponentially stable.

The solution to (7.1) and the solution to (7.3) are related by a Lyapunov transformation, namely $L(t) := \Phi_A(t,t_0)e_R^{-1}(t,t_0)$. By Theorem 3.7 and Corollary 3.8, the solution to the system (7.1) is exponentially stable if and only if the solution to the system (7.3) is exponentially stable.

Case (2): Suppose $\mathrm{Re}_\mu(\gamma_k(t)) = 0$ for some $1 \leqslant k \leqslant n$ with equal algebraic and geometric multiplicities corresponding to $\gamma_k(t)$. Then the Jordan block corresponding to $\gamma_k(t)$ is 1×1, which implies that the mode vectors have the form $m_k(t) = \beta_k e_{\gamma_k}(t,t_0)$.

Thus,

$$\lim_{t\to\infty}\|m_k(t)\| \leqslant \lim_{t\to\infty} \beta_k\left|e_{\gamma_k}(t,t_0)\right| \leqslant \lim_{t\to\infty} \beta_k e_{\mathrm{Re}_\mu(\gamma_k)}(t,t_0) = \beta_k < \infty.$$

By Theorem 7.5 (2), the system (7.3) is stable. Thus, since (7.1) is related to (7.3) by a Lyapunov transformation, (7.1) is stable.

Now suppose that for some k the algebraic and geometric multiplicities corresponding to $\gamma_k(t)$ are not equal. Then, the elements in the corresponding mode vector are made up of generalized polynomials. Thus, by Theorem 7.6, since there is no decaying exponential function to control the growth of these polynomials, solutions grow at the rate of the polynomials. Therefore, (7.3) is unstable, implying by the Lyapunov transformation that (7.1) is also unstable.

Case (3): Suppose that for some $1 \leqslant i \leqslant n$, we have $\mathrm{Re}_\mu(\gamma_i(t)) > 0$. This implies

$$\lim_{t\to\infty}\|e_R(t,t_0)\| = \infty,$$

thus implying by the Lyapunov transformation

$$\lim_{t\to\infty}\|\Phi_A(t,t_0)\| = \infty. \qquad \square$$

The consequences of Theorem 7.9 should be underscored at this point: from the construction of the corresponding system matrix $R(t)$, we can deduce the stability of (7.3) solely on the placement of the traditional eigenvalues in the complex plane. In general, this is not true of time varying systems. However, we have shown that for this unified Floquet theory, the associated (and in general time varying) matrix $R(t)$ yields stability characteristics through simple pole placement in the complex plane—exactly like constant systems and Jordan reducible time varying systems on $\mathbb{R}$ or $h\mathbb{Z}$. Moreover, the matrix $R(t)$ is in fact constant if and only if the domain of the underlying dynamical system has constant graininess, e.g. $\mathbb{R}$ or $h\mathbb{Z}$. The upshot of course is that this Floquet theory extends to any domain that is a p-periodic closed subset of $\mathbb{R}$, including those that have nonconstant discrete points as well as mixed continuous and discrete intervals.

A key observation about this unified Floquet theory is that the associated system matrix $R(t)$ yields the stability characteristics of the original system by placement of the traditional eigenvalues alone—although it accomplishes this via an associated time varying systems rather than with a time invariant system.

The next result is a direct consequence of Theorem 7.9.

Corollary 7.10. *Consider the p-periodic system* (3.2).

(1) *If all the Floquet multipliers have modulus less than one, then the system* (3.2) *is exponentially stable.*
(2) *If all of the Floquet multipliers have modulus less than or equal to one, and if, for each Floquet multiplier with modulus equal to one, the algebraic multiplicity equals the geometry multiplicity, then the system* (3.2) *is stable*; *otherwise the system* (3.2) *is unstable, growing at rates of generalized polynomials of t.*
(3) *If at least one Floquet multiplier has modulus greater than one, then the system* (3.2) *is unstable.*

In light of Corollary 7.10, we can make the following conclusions about the examples in Section 6. Solutions of (4.1) and (4.7) are exponentially stable since the Floquet multipliers satisfy $|\lambda_i| < 1$, $i = 1, 2$. Solutions of (4.4) are uniformly stable since the Floquet multipliers satisfy $|\lambda_i| \leqslant 1$, $i = 1, 2$ and their algebraic and geometric multiplicities are equal.

Remark. We can think of an eigenvalue $\gamma(t) = \lim_{s \searrow \mu(t)} \frac{\lambda^{\frac{s}{p}} - 1}{s}$ of $R(t)$ as constant with respect to its "placement" in the instantaneous stability region corresponding to the element t in the p-periodic time scale $\mathbb{T}$. In other words, the exponential bound is independent of $\mu(t)$:

$$\begin{aligned} e_\gamma(t, t_0) &= \exp\left[\int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{1}{s} \operatorname{Log}\bigl(1 + s\gamma(\tau)\bigr)\Delta\tau\right] \\ &= \exp\left[\int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{1}{s} \operatorname{Log}\left(1 + s\frac{\lambda^{\frac{s}{p}} - 1}{s}\right)\Delta\tau\right] \\ &= \lambda^{\frac{t - t_0}{p}}. \end{aligned}$$

8. Conclusions

A unified and extended Floquet theory has been developed, which includes a canonical Floquet decomposition in terms of a generalized matrix exponential function. The notion of time varying Floquet exponents and their relationship to constant Floquet multipliers has been introduced. We presented a closed-form answer to the open problem of finding a solution matrix $R(t)$ to the equation $e_R(t, \tau) = M$ on an arbitrary time scale $\mathbb{T}$, where $R(t)$ and M are $n \times n$ matrices, and M is constant and nonsingular. The development of this Floquet theory on both homogeneous and nonhomogeneous hybrid dynamical systems was completed by developing Lyapunov transformations and a stability analysis of the original system in terms of a corresponding (but not necessarily autonomous) system that, by its construction, yields stability characteristics by traditional eigenvalue placement in the complex plane.

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Appendix A. Real powers of a matrix

We introduce the technical preliminaries required to define the real power of a matrix, which is of paramount importance in the derivation of the unified Floquet theory. The following can be found in [17,18].

Definition A.1. Given an $n \times n$ invertible matrix M with elementary divisors $\{(\lambda - \lambda_i)^{m_i}\}_{i=1}^{k}$, we let the characteristic polynomial be the function $p(\lambda)$. We define the polynomial $a_i(\lambda)$ implicitly via the expansion

$$\frac{1}{p(\lambda)} = \sum_{i=1}^{k} \frac{a_i(\lambda)}{(\lambda - \lambda_i)^{m_i}}.$$

The polynomial $b_i(\lambda)$ is defined by omitting the factor $(\lambda - \lambda_i)^{m_i}$ from the characteristic polynomial $p(\lambda)$, that is

$$b_i(\lambda) := \prod_{j \neq i} (\lambda - \lambda_j)^{m_j}.$$

Then the *ith projection matrix* is defined as

$$P_i(\lambda) := a_i(\lambda) b_i(\lambda). \tag{A.1}$$

A property of the set of the projection matrices is the following. By definition,

$$1 = \sum_{j=1}^{k} \frac{a_j(\lambda)}{(\lambda - \lambda_j)^{m_j}} p(\lambda) = \sum_{j=1}^{k} a_j(\lambda) b_j(\lambda) = \sum_{j=1}^{k} P_j(\lambda). \tag{A.2}$$

When the scalar value λ is replaced by the $n \times n$ nonsingular matrix M, (A.2) becomes

$$I = \sum_{j=1}^{k} P_j(M). \tag{A.3}$$

We now state and prove three propositions concerning the projection matrices.

Proposition A.2. *Given an $n \times n$ nonsingular matrix M as in Definition* A.1, $P_i(M)(M - \lambda_i I)^{m_i} = 0$.

Proof. Observe

$$P_i(M)(M - \lambda_i I)^{m_i} = a_i(M) b_i(M)(M - \lambda_i I)^{m_i} = a_i(M) \prod_{j=1}^{k} (M - \lambda_j I)^{m_j} = a_i(M) p(M) = 0. \qquad \square$$

Proposition A.3. *The set $\{P_i(M)\}_{i=1}^{k}$ is orthogonal. That is,* $P_i(M) P_j(M) = \delta_{ij} P_i(M)$.

Proof. Suppose $i \neq j$. Then

$$\begin{aligned} P_i(M) P_j(M) &= P_i(M) a_j(M) b_j(M) \\ &= P_i(M) a_j(M) \prod_{k \neq j} (M - \lambda_k I)^{m_k} \end{aligned}$$

$$= P_i(M)a_j(M)(M-\lambda_i I)^{m_i} \prod_{k\neq i,j}(M-\lambda_k I)^{m_k}$$

$$= P_i(M)(M-\lambda_i I)^{m_i} a_j(M) \prod_{k\neq i,j}(M-\lambda_k I)^{m_k}$$

$$= 0.$$

To conclude the proof, using the property (A.3) and multiplying on each side by the ith projection matrix,

$$P_i(M) = \sum_{j=1}^{k} P_i(M)P_j(M) = P_i(M)P_i(M). \qquad \square$$

Proposition A.4. *Given an $n \times n$ nonsingular matrix M as in Definition* A.1 *with corresponding Jordan canonical form $J = CMC^{-1}$, for some nonsingular matrix C,*

$$CP_i(M)C^{-1} = P_i(J),$$

for all $1 \leqslant i \leqslant k$.

Proof. For any $1 \leqslant i \leqslant k$, by definition of the projection matrix (A.1), we see that $P_i(M)$ is the product of two polynomials of the matrix M. Further, by definition of a_i and b_i, it follows that $Ca_i(M)C^{-1} = a_i(J)$ and $Cb_i(M)C^{-1} = b_i(J)$. Thus,

$$CP_i(M)C^{-1} = Ca_i(M)b_i(M)C^{-1} = Ca_i(M)C^{-1}Cb_i(M)C^{-1} = a_i(J)b_i(J) = P_i(J). \qquad \square$$

We are now in the position to state the definition of the principal value of the real power of a nonsingular $n \times n$ matrix.

Definition A.5. Given an $n \times n$ nonsingular matrix M as in Definition A.1 and any $r \in \mathbb{R}$, we define the real power of the matrix M by

$$M^r := \sum_{i=1}^{k} P_i(M)\lambda_i^r \left[\sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \left(\frac{M-\lambda_i I}{\lambda_i}\right)^j\right].$$

The next theorem gives a relationship between the real power of a matrix and the real power of the corresponding Jordan canonical form.

Theorem A.6. *Given an $n \times n$ nonsingular matrix M as in Definition* A.1 *with corresponding Jordan canonical form $J = CMC^{-1}$, for an appropriately defined nonsingular matrix C, and any $r \in \mathbb{R}$,*

$$CM^rC^{-1} = J^r.$$

Proof. The result follows from a direct application of Proposition A.4 and Definition A.5. Observe,

$$CM^rC^{-1} = C\sum_{i=1}^{k} P_i(M)\lambda_i^r \left[\sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \left(\frac{M-\lambda_i I}{\lambda_i}\right)^j\right]C^{-1}$$

$$= \sum_{i=1}^{k} CP_i(M)C^{-1}C\lambda_i^r \left[\sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \left(\frac{M-\lambda_i I}{\lambda_i}\right)^j\right]C^{-1}$$

$$= \sum_{i=1}^{k} P_i(J)\lambda_i^r \left[\sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \left(\frac{J-\lambda_i I}{\lambda_i} \right)^j \right]$$
$$= J^r. \qquad \square$$

Remark. Theorem A.6 is a generalization from the familiar matrix property that $CM^kC^{-1} = J^k$, for all $k \in \mathbb{N}_0$.

Remark. Given any nonsingular $n \times n$ matrix M as in Definition A.1, we have $P_i(M)(M-\lambda_i I)^{m_i} = 0$, for $i = 1, \ldots, k$ and $m_1 + \cdots + m_k = n$. However, since $P_i(M)(M-\lambda_i I)$ is nilpotent by Propositions A.2 and A.3, there exists an integer $n_i \leqslant m_i$ such that $P_i(M)(M-\lambda_i I)^{n_i-1} \neq 0$, but $P_i(M)(M-\lambda_i I)^{n_i} = 0$. Then $P_i(M)(M-\lambda_i I)^{n_i}$ is a minimal polynomial of M.

Theorem A.7. *A nonzero column vector v_r of $P_i(M)(M-\lambda_i I)^{n_i-r}$ is a generalized eigenvector of rank r, for $1 \leqslant r \leqslant n_i$, associated with the eigenvalue λ_i. A regular eigenvector has rank $r = 1$.*

Proof. For $r = 1$,

$$MP_i(M)(M-\lambda_i I)^{n_i-1} = \sum_{j=1}^{k} P_j(M)\big[\lambda_j I + (M-\lambda_j I)\big]P_i(M)(M-\lambda_i I)^{n_i-1}$$
$$= \lambda_i P_i(M)(M-\lambda_i I)^{n_i-1},$$

where we use the result from Proposition A.3 which states that $P_i(M)P_j(M) = \delta_{ij}P_i(M)$, for all $1 \leqslant i, j \leqslant k$. Thus, since $P_i(M)(M-\lambda_i I)^{n_i-1}$ has rank one, we choose a nonzero $n \times 1$ column vector v_1 from this matrix and obtain

$$Mv_1 = \lambda_i v_1.$$

When $r > 1$, note that

$$P_i(M)(M-\lambda_i I)^{n_i-r} = (M-\lambda_i I)P_i(M)(M-\lambda_i I)^{n_i-r-1},$$

which implies

$$v_r = (M-\lambda_i I)v_{r+1}. \qquad \square$$

The following theorem will show that for any invertible $n \times n$ matrix M, with d eigenpairs $\{\lambda_i, v_i\}_{i=1}^d$, and any $r \in \mathbb{R}$, the matrix M^r has the d eigenpairs $\{\lambda_i^r, v_i\}_{i=1}^d$.

Theorem A.8. *Given an invertible $n \times n$ matrix M as in Definition* A.1, *with d eigenpairs $\{\lambda_i, v_i\}_{i=1}^d$, and any $r > 0$, for each eigenpair $\{\lambda_i, v_i\}$ of M, we have $M^r v_i = \lambda_i^r v_i$.*

Proof. Let $1 \leqslant s \leqslant d$ and λ_s be an eigenvalue of the matrix M. Then, noting that $(M-\lambda_s I)^{n_s}$ is a minimal polynomial of M, using Proposition A.2 and the remark before Theorem A.7,

$$M^r P_s(M)(M-\lambda_s I)^{n_s-1} = \left[\sum_{i=1}^{d} P_i(M)\lambda_i^r \left[\sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \left(\frac{M-\lambda_i I}{\lambda_i} \right)^j \right] \right] P_s(M)(M-\lambda_s I)^{n_s-1}$$

$$= P_s(M)\lambda_s^r \sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \lambda_i^{-j} (M-\lambda_i I)^{j+n_s-1}$$

$$= \lambda_s^r P_s(M)(M-\lambda_s I)^{n_s-1},$$

for each nonzero column vector v_s of $P_s(M)(M-\lambda_s I)^{n_s-1}$. The matrix $P_s(M)(M-\lambda_s I)^{n_s-1}$ has rank 1 and we can take any nonzero column vector v_s from this matrix as an eigenvector corresponding the eigenvalue λ_s of the matrix M. This vector is a linear combination of all of the eigenvectors associated with the eigenvalue λ_s. Thus, $M^r v_s = \lambda_s^r v_s$. □

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