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State Feedback Stabilization of Linear Time-Varying Systems on Time Scales

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***Abstract*— A fundamental result in linear system theory is the development of a linear state feedback stabilizer for time-varying systems under suitable controllability constraints. This result was previously restricted to systems operating on the continuous ($\mathbb{R}$) and uniform discrete ($h\mathbb{Z}$) time domains with constant step size h. Using the framework of dynamic equations on time scales, we construct a linear state feedback stabilizer for time-varying systems on arbitrary time domains[1].**

I. Introduction

Linear systems theory is well-studied in both the continuous and discrete settings [2], [10], [33], i.e. settings in which the independent variable t (often representing time) is constrained to $t \in \mathbb{R}$ or $t \in h\mathbb{Z}$ with h fixed. Recently, however, attention has turned to generalizing the theories on $\mathbb{R}$ and $\mathbb{Z}$ to nonuniform discrete domains or domains with a mixture of discrete and continuous parts. Progress toward this has been made on the topics of controllability/observability and reachability/realizability [15], [26], Laplace transforms [16], [17], [18], Fourier transforms [29], Lyapunov equations [13], [19], and various types of stability results including Lyapunov, exponential, and BIBO [6], [13], [26]. The goal is not to simply reprove existing, well-known theories, but rather to view $\mathbb{R}$ and $\mathbb{Z}$ as special cases of a single, overarching theory and to extend the theory to dynamical and control systems on these more general domains. Doing so reveals a rich mathematical structure which has great potential for new applications in diverse areas such as adaptive control [20], real-time communications networks [21], [22], dynamic programming [34], switched systems [30], stochastic models [5], population models [37], and economics [3], [4]. The focus of this paper is the study of linear state feedback controllers [28], [35] in this generalized setting and to compare and contrast these results with the standard continuous and uniform discrete scenarios.

The fast-growing field of *dynamic equations on time scales* (DETS) provides the mathematical foundation for what follows, including a calculus that is based on the notion of the dynamic (or "Hilger") derivative. A short introduction to time scales appears in the Appendix; for a more comprehensive introduction, readers are referred to two texts [7], [8]. Henceforth, a working understanding of DETS is assumed. It is not possible to include the full text of every proof in this paper; however, these results and others are expanded in complete detail elsewhere [27].

The paper begins with some definitions, including the controllability Grammian and the weighted-controllability Grammian. After establishing some important properties of the Grammian, the main theorem postulates a feedback control law that can uniformly exponentially stabilize time-varying closed loop systems with arbitrary rate. The paper concludes with some straightforward experimental results that illustrate an application of the main theorem.

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II. State Feedback Stabilization

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be rd-continuous on $\mathbb{T}$ with $p, m \le n$, and consider the open-loop state equation

$$x^{\Delta}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \tag{1}$$

In the presence of a linear state feedback controller, we replace the input $u(t)$ above with $u(t) := K(t)x(t) + N(t)r(t)$, where $r(t)$ represents a new input signal, and $K(t) \in \mathbb{R}^{m\times n}$, $N(t) \in R^{m\times m}$ are rd-continuous. The corresponding closed-loop system is

$$\begin{aligned} x^{\Delta}(t) &= [A(t) + B(t)K(t)]\,x(t) + B(t)N(t)r(t), \\ x(t_0) &= x_0, \end{aligned} \tag{2}$$

Without loss of generality, we proceed with $r(t) \equiv 0$.

Definition 1: [15] Let $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ both be rd-continuous functions on $\mathbb{T}$, with $p, m \le n$. The regressive linear system

$$x^{\Delta}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \tag{3}$$

is *controllable* on $[t_0, t_f]$ if given any initial state x_0 there exists a rd-continuous input signal $u(t)$ such that the corresponding solution of the system satisfies $x(t_f) = x_f$.

From Davis et al. [15] it is known that invertibility of the Gramiam matrix

$$\mathcal{G}_C(t_0, t_f) := \int_{t_0}^{t_f} \Phi_A(t_0, \sigma(t))B(t)B^T(t)\Phi_A^T(t_0, \sigma(t))\,\Delta t, \tag{4}$$

where $\Phi_Z(t, t_0)$ is the transition matrix for the system $X^{\Delta}(t) = Z(t)X(t)$, $X(t_0) = I$, is a necessary and sufficient condition to ensure controllability of (3).

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Unlike on $\mathbb{R}$ or $\mathbb{Z}$, in the general time scales setting, there are various ways one could legitimately define *exponential stability*. Pötzsche, Siegmund, and Wirth [32] first did so by bounding the state vector above by a decaying *regular* exponential function. In this paper, we adopt the definition of DaCunha, [13] who generalized their definition by allowing the state vector to be bounded above by a time scale exponential function of the form $e_{-\lambda}(t, t_0)$, with $-\lambda \in \mathcal{R}^+$.

Definition 2: [13] The regressive linear state equation (3) is *uniformly exponentially stable* with rate $\lambda > 0$, where $-\lambda \in \mathcal{R}^+$, if there exists a constant $\gamma > 0$ such that for any $t_0 \in \mathbb{T}$ and x_0 the corresponding solution satisfies

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)\|x_0\|, \qquad t \geq t_0. \tag{5}$$

Before moving to the main result of the paper, a theorem is needed. Its proof is omitted for space, but it essentially follows from a generalization of Lyapunov's direct method to time scale settings. We note here that, while Lyapunov techniques were employed in the narrative that follows, results in the literature suggest that direct examination of the solutions to time-varying linear dynamical systems may yield insight as well [36].

Theorem 3: Suppose $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. The regressive time varying linear dynamic system

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{6}$$

is uniformly exponentially stable if there exists a symmetric matrix $Q(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ such that for all $t \in \mathbb{T}$

(i) $\eta I \leq Q(t) \leq \rho I$,

(ii) $\left[(I + \mu(t)A^T(t))Q(\sigma(t))(I + \mu(t)A(t)) - Q(t)\right]/\mu(t) \leq -\nu I$, where $\nu, \eta, \rho > 0$ and $-\frac{\nu}{\rho} \in \mathcal{R}^+$.

In order to achieve the desired stabilization result, we first define a weighted version of the controllability Gramian. For $\alpha > 0$ define the *α-weighted controllability Gramian matrix* $\mathcal{G}_{C_\alpha}(t_0, t_f)$ by

$$\begin{aligned}\mathcal{G}_{C_\alpha}(t_0, t_f) \quad : \quad = \int_{t_0}^{t_f} (e_\alpha(t_0, s))^4 \Phi_A(t_0, \sigma(s))B(s)... \\ B^T(s)\Phi_A^T(t_0, \sigma(s))\Delta s.\end{aligned} \tag{7}$$

With this, we are now in position to show the main result of the paper.

Theorem 4 (Gramian Exponential Stability Criterion): Consider the regressive linear state equation of (3) on a time scale $\mathbb{T}$ such that $\mu_{\min} \leq \mu(t) \leq \mu_{\max}$ for all $t \in \mathbb{T}$. Suppose there exist constants $\varepsilon_1, \varepsilon_2 > 0$ and a strictly increasing function $\mathcal{C} : \mathbb{T} \to \mathbb{T}$ such that $0 < \mathcal{C}(t) - t \leq M$ holds for some constant $0 < M < \infty$ and all $t \in \mathbb{T}$ with

$$\varepsilon_1 I \leq \mathcal{G}_C(t, \mathcal{C}(t)) \leq \varepsilon_2 I, \quad \text{for all } t \in \mathbb{T}. \tag{8}$$

Then given $\alpha > 0$, the state feedback gain

$$K(t) := -B^T(t)(I + \mu(t)A^T(t))^{-1}\mathcal{G}^{-1}_{C_\alpha}(t, \mathcal{C}(t)), \tag{9}$$

has the property that the resulting closed-loop state equation is uniformly exponentially stable with rate α. We call $\mathcal{C}(t)$ the *controllability window* for the problem.

The essential arguments of the proof follow. We first note that for $N = \sup_{t\in\mathbb{T}} \frac{\log(1+\mu(t)\alpha)}{\mu(t)}$, we have $0 < N < \infty$ since $\mathbb{T}$ has bounded graininess. Thus,

$$\begin{aligned} e_\alpha(t, \mathcal{C}(t)) &= \exp\left(-\int_t^{\mathcal{C}(t)} \frac{\log(1 + \mu(s)\alpha)}{\mu(s)}\Delta s\right) \\ &\geq \exp\left(-\int_t^{\mathcal{C}(t)} N\Delta s\right) \\ &= e^{-N(\mathcal{C}(t)-t)} \\ &\geq e^{-MN}. \end{aligned} \tag{10}$$

Comparing the quadratic forms $x^T\mathcal{G}_C(t, \mathcal{C}(t))x$ and $x^T\mathcal{G}_C(t, \mathcal{C}(t))x$ gives

$$\begin{aligned} e^{-4MN}\mathcal{G}_C(t, \mathcal{C}(t)) \quad &\leq \quad \mathcal{G}_{C_\alpha}(t, \mathcal{C}(t)) \leq \mathcal{G}_C(t, \mathcal{C}(t)), \\ \text{for all } t \quad &\in \quad \mathbb{T}. \end{aligned} \tag{11}$$

Thus, (8) implies

$$\varepsilon_1 e^{-4MN} I \leq \mathcal{G}_{C_\alpha}(t, \mathcal{C}(t)) \leq \varepsilon_2 I, \text{for all } t \in \mathbb{T}, \tag{12}$$

and so the existence of $\mathcal{G}^{-1}_{C_\alpha}(t, \mathcal{C}(t))$ is immediate.

Next we define $\hat{A}(t)$ to be the system matrix under the closed loop feedback law of (9), i.e.

$$\hat{A}(t) := A(t) - B(t)B^T(t)(I + \mu(t)A^T(t))\mathcal{G}^{-1}_{C_\alpha}(t, \mathcal{C}(t)). \tag{13}$$

Rather than attempting to analyze the stability of $x^{\Delta}(t) = \hat{A}(t)x(t)$ directly, it is more straightforward to show the uniform exponential stability of

$$z^{\Delta}(t) = [\hat{A}(t)(1 + \mu(t)\alpha) + \alpha I]z(t), \tag{14}$$

and then induce stability of $x^{\Delta}(t) = \hat{A}(t)x(t)$ by noting that (14) follows immediately from a change of variables, $z(t) = e_\alpha(t, t_0)x(t)$. Since $\alpha > 0$, if $z(t)$ is exponentially bounded, then $x(t)$ must be as well.

The uniform exponential stability of (14) follows from application Theorem 3 with $Q(t) = \mathcal{G}^{-1}_{C_\alpha}$ and $A(t)$ replaced by $[\hat{A}(t)(1+\mu(t)\alpha)+\alpha I]$. Requirement (i) of the theorem follows immediately from (11) and the fact that $Q(t)$ is symmetric and continuously differentiable. Requirement (ii), namely that there exists a $\nu > 0$ with $-\frac{\nu}{\rho} \in \mathcal{R}^+$, is more difficult to establish; readers are referred Jackson, et al. [27].

Given that (14) is uniformly exponentially stable, z must have some bounding rate $\lambda > 0$. From the change of variables, then, it follows that x is stable with rate at least α, from which follows the uniform exponential stability of (3). This concludes the exposition of Theorem 4.

III. The Controllability Window

It is natural to wonder about the controllability window defined in Theorem 4. In particular, is it too restrictive to assume that $\mathcal{C}(t)$ always exists, and what form might it assume?

First, the assumed bound (8) is not overly restrictive – it is merely a reformulation of the controllability Gramian invertibility criterion, and controllability of the open-loop system is a prerequisite for feedback stabilization. The requirement that $\mathcal{C}(t)$ be increasing on an interval just ensures a nondegenerate interval (in the time scale) on which the open-loop system is controllable.

One possible construction of $\mathcal{C}(t)$ is, for any $\delta_1, \delta_2 > 0$,

$$\mathcal{C}(t) := \begin{cases} t + \delta_1, & \text{if } \sigma(t) = t, \\ \sigma^k(t), & \text{if } \sigma^i(t) \neq t \text{ for all } 0 \leq i \leq k, \\ \sigma^k(t) + \delta_2, & \text{else,} \end{cases} \quad (15)$$

where σ^k means the composition of the forward jump operator σ with itself $k-1$ times.

Note that, for $\mathbb{T} = \mathbb{R}$, $\mathcal{C}(t) = t+\delta$ for any $\delta > 0$ is sufficient, while on $\mathbb{T} = \mathbb{Z}$, the function $\mathcal{C}(t) = t + k$ for $k \in \mathbb{N}$ meets the criteria. These coincide with the controllability windows found in the literature for both the continuous and discrete cases [2], [10], [33], a very satisfying result.

IV. Experimental Results

Throughout the preceding discussion, it has been assumed that the time scale is known *a priori*; in other words, that a system's time domain is known before the system dynamics "start" at time $t = 0$. Under this assumption, it is possible to calculate feedback gain $K(t)$ *a priori* if the system's state matrices $A(t)$ and $B(t)$ are known. Scenarios in which non-standard time scales (not $\mathbb{R}$ or $h\mathbb{Z}$) are useful may come about for different reasons. For example, it may be that a computer controller cannot guarantee consistent hard deadlines (i.e. "real time" response) for communication with sensors and actuators; in this case, a time scale may be scheduled that is more amenable to the other tasks the computer is performing. A similar problem may occur in a networked, or distributed, control system, in which various network traffic activities determine the time scale (e.g. packet arrival times are Poisson-distributed). In any case, if the time scale is known, or at least known over the controllability window, the feedback gain may be computed and applied in advance.

To illustrate the paper's central theorem in hardware, a simple experiment was devised using a DC motor with an inertial mass. A system identification procedure produced approximate 2^{nd}-order state matrices

$$\begin{aligned} \hat{A} &= \begin{bmatrix} 0 & 1 \\ 0 & -0.15 \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} 0 \\ 13.8 \end{bmatrix}, \\ \frac{d\hat{x}(t)}{dt} &= \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t), \qquad t \in \mathbb{R}, \end{aligned} \quad (16)$$

where state vector $\hat{x}(t)$ is the motor's angular shaft position (rev) and velocity (rev/s), and $\hat{u}(t)$ is the input voltage (V). Electrical dynamics were neglected due to the relatively small electrical time constant. The hat notation designates $\hat{A}$ and $\hat{B}$ as the state matrices of a dynamical system on $\mathbb{R}$. Sample-and-hold discretization to an arbitrary time scale $\mathbb{T}$ gives

$$\begin{aligned} A(t) &= \left[\frac{e^{\hat{A}\mu(t)} - I}{\mu(t)}\right], \; B(t) = \left[\sum_{i=1}^{\infty} \frac{(\hat{A}\mu(t))^{i-1}}{i!}\right]\hat{B}, \\ t &\in \mathbb{T}. \end{aligned} \quad (17)$$

Now equation (3) is in force.

To begin, several discrete time scales $\mathbb{T}$ were selected and populated with anywhere from $\ell = 20$ to 100 points (all of the time scales in these experiments were purely discrete with no continuous intervals). Choosing a window operator $\mathcal{C}(t) = \sigma^k(t)$ simply amounted to choosing a window size $k > 0$. Some ramifications of this choice are discussed later. Next, using MATLAB, $K(t)$ was computed over the first $\ell - k$ points in the time scale. $K(t)$ and $\mathbb{T}$, along with control law $u(t) = K(t)x(t) + N(t)r(t)$ where $N(t) \equiv 1$ and $r(t) = 2h(t)$ where h denotes a unit step function, were programmed into a computer running the QNX real-time operating system and outfitted with digital and analog input/output cards. Internal high-precision timers were employed so that the system would acquire the motor states $x(t)$, and apply drive current $u(t)$, only at the pre-determined points in $t \in \mathbb{T}$. The resulting state trajectories therefore illustrate the closed-loop system step response of the motor and mass.

Three examples of the closed-loop step response are shown in Figure 1. Time scale $\mathbb{T}_a$ of example (a) was created with widely varying graininess. Graininess $\mu_a(t)$ occurs in multiples of 10ms, with the first four points exhibiting graininess of 80 or 90ms, and points thereafter exhibiting graininess of 10 or 20ms. This time scale was designed to emulate the timing of a real-time process that is unable to meet hard 10ms deadlines. If a deadline is missed, i.e. the controller cannot respond at the next specified $t \in \mathbb{T}_a$, the next point in the time scale is scheduled some multiple of 10ms in the future. Time scale $\mathbb{T}_b$ of example (b) exhibits graininess from a uniformly random distribution between 80 and 150ms. The third example (c) combines two interesting phenomena: a time scale $\mathbb{T}_c$ of uniformly random distribution, with a very large gap in the middle. Example (c) is particularly interesting because it can be seen that the controller has computed its best estimate (as close as the model allows) of the open-loop constant input current required to move the motor shaft to near-zero error by the end of the gap.

In each example, it was necessary to choose a window size k as well as a constant α for the computation of $K(t)$. The performance impact of different choices is not obvious. Small controllability windows would seem to induce better performance in terms of settling time; however, they also induce large input magnitudes that may exceed the physical limitations of the system. The effects of k and α are explored further in [27].

V. Conclusion

The paper's main theorem, Theorom , and the experimental examples illustrate that the full-state, closed-loop feedback $u(t) = K(t)x(t)$ will indeed stabilize time-varying linear dynamical systems on a variety of interesting time scales. However, there are several practical limitations to overcome. First, the actual computation of $K(t)$ is very complex; in the experimental trials, it was computed beforehand and uploaded to the real-time controller. Second, $K(t)$ depends on knowledge of the time scale over some finite future window (defined

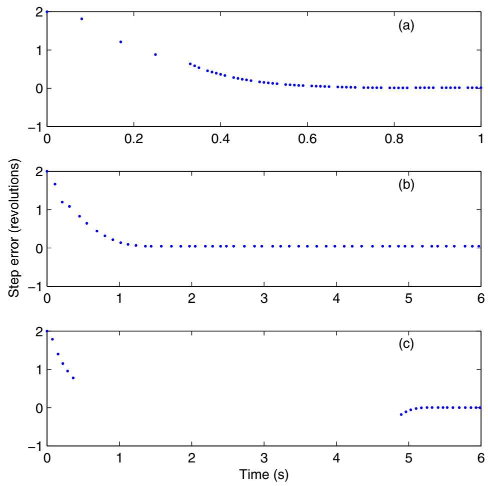

Fig. 1. Step responses for the three cases discussed in the text. Note that the time axis in case (a) has been magnified in order to see the individual points in the time scale.

by the operator $\mathcal{C}(t)$). Thus, K is not strictly causal (although it does not depend on knowledge of system states other than time in the future). Third, K depends on knowledge of the system parameters, which are often not well known. It should be noted, however, that the first and third of these limitations also apply in the classical cases as well (feedback control on $\mathbb{R}$ and $\mathbb{Z}$). The second limitation is obviated on $\mathbb{R}$ and $\mathbb{Z}$ because the time scale is always known *a priori*.

VI. Appendix: Dynamic Equations on Time Scales

A. What Are Time Scales?

The theory of time scales springs from the 1988 doctoral dissertation of Stefan Hilger [25] that resulted in his seminal paper [24]. These works aimed to unify various overarching concepts from the (sometimes disparate) theories of discrete and continuous dynamical systems [31], but also to extend these theories to more general classes of dynamical systems. From there, time scales theory advanced fairly quickly, culminating in the excellent introductory text by Bohner and Peterson [8] and the more advanced monograph [7]. A succinct survey on time scales can be found in [1].

A *time scale* $\mathbb{T}$ is any nonempty, (topologically) closed subset of the real numbers $\mathbb{R}$. Thus time scales can be (but are not limited to) any of the usual integer subsets (e.g. $\mathbb{Z}$ or $\mathbb{N}$), the entire real line $\mathbb{R}$, or any combination of discrete points unioned with closed intervals. For example, if $q > 1$ is fixed, the *quantum time scale* $\overline{q^{\mathbb{Z}}}$ is defined as

$$\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}.$$

The quantum time scale appears throughout the mathematical physics literature, where the dynamical systems of interest are the q-difference equations [9],[11]. Another interesting example is the *pulse time scale* $\mathbb{P}_{a,b}$ formed by a union of closed intervals each of length a and gap b:

$$\mathbb{P}_{a,b} := \bigcup_k \left[k(a+b), k(a+b)+a\right].$$

Other examples of interesting time scales include any collection of discrete points sampled from a probability distribution, any sequence of partial sums from a series with positive terms, or even the famous Cantor set.

The bulk of engineering systems theory to date rests on two time scales, $\mathbb{R}$ and $\mathbb{Z}$ (or more generally $h\mathbb{Z}$, meaning discrete points separated by distance h). However, there are occasions when necessity or convenience dictates the use of an alternate time scale. The question of how to approach the study of dynamical systems on time scales then becomes relevant, and in fact the majority of research on time scales so far has focused on expanding and generalizing the vast suite of tools available to the differential and difference equation theorist. We now briefly outline the portions of the time scales theory that are needed for this paper to be as self-contained as is practically possible.

B. The Time Scales Calculus

The *forward jump operator* is given by $\sigma(t) := \inf_{s\in\mathbb{T}}\{s > t\}$, while the *backward jump operator* is $\rho(t) := \sup_{s\in\mathbb{T}}\{s < t\}$. The *graininess function* $\mu(t)$ is given by $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. A point $t \in \mathbb{T}$ is *left-scattered* if $\rho(t) < t$ and *left dense* if $\rho(t) = t$. If t is both left-scattered and right-scattered, we say t is *isolated* or *discrete*. If t is both left-dense and right-dense, we say t is *dense*. The set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, define $f^\Delta(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U. \tag{18}$$

The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the *delta derivative* or the *Hilger derivative* of f on $\mathbb{T}^\kappa$. Equivalently, (18) can be restated to define the Δ-differential operator as

$$x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)},$$

where the quotient is taken in the sense that $\mu(t) \to 0^+$ when $\mu(t) = 0$.

A benefit of this general approach is that the realms of differential equations and difference equations can now be viewed as but special, particular cases of more general *dynamic equations on time scales*, i.e. equations involving the delta derivative(s) of some unknown function. See Table I.

Naturally, with any discussion of derivatives a notion of "continuity" is required. For $f : \mathbb{T} \to \mathbb{X}$, the function f is said to be right-dense continuous, or *rd-continuous*, if it is continuous (in the usual sense) over any right-dense interval

TABLE I

DIFFERENTIAL OPERATORS ON TIME SCALES.

time scale	differential operator	notes	integral operator	notes
$\mathbb{T}$	$x^{\Delta}(t) = \frac{x(\sigma(t))-x(t)}{\mu(t)}$	generalized derivative	$\int_a^b f(t)\,\Delta t$	generalized integral
$\mathbb{R}$	$x^{\Delta}(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$	standard derivative	$\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$	standard Lebesgue integral
$\mathbb{Z}$	$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t)$	forward difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$	summation operator
$h\mathbb{Z}$	$x^{\Delta}(t) = \Delta_h x(t) := \frac{x(t+h)-x(t)}{h}$	h-forward difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-h} f(t)h$	h-summation
$\overline{q^{\mathbb{Z}}}$	$x^{\Delta}(t) = \Delta_q x(t) := \frac{x(qt)-x(t)}{(q-1)t}$	q-difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b/q} \frac{f(t)}{(q-1)t}$	q-summation
$\mathbb{P}_{a,b}$	$x^{\Delta}(t) = \begin{cases} \frac{dx}{dt}, & \sigma(t) = t, \\ \frac{x(t+b)-x(t)}{b}, & \sigma(t) > t \end{cases}$	pulse derivative		

within $\mathbb{T}$. The set of all rd-continuous functions that are n-times differentiable is denoted $C^n_{rd}(\mathbb{T}, \mathbb{X})$.

Since the graininess function induces a measure on $\mathbb{T}$, if we consider the Lebesgue integral over $\mathbb{T}$ with respect to the μ-induced measure,

$$\int_{\mathbb{T}} f(t)\, d\mu(t),$$

then all of the standard results from measure theory are available [23]. The upshot is that the derivative and integral concepts apply just as readily to *any* closed subset of the real line as they do on $\mathbb{R}$ or $\mathbb{Z}$; see Table 1. Our goal is to leverage this general framework against wide classes of dynamical and control systems.

The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. We define the related sets

$$\mathcal{R} := \{p : \mathbb{T} \to \mathbb{R} : p \in C_{rd}(\mathbb{T}) \text{ and } 1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^{\kappa}\},$$
$$\mathcal{R}^+ := \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^{\kappa}\}.$$

For $p(t) \in \mathcal{R}$, we define the *generalized time scale exponential function* $e_p(t, t_0)$ as the unique solution to the initial value problem $x^{\Delta}(t) = p(t)x(t)$, $x(t_0) = 1$, which exists when $p \in \mathcal{R}$. See [7].

Similarly, the unique solution to the matrix initial value problem $X^{\Delta}(t) = A(t)X(t)$, $X(t_0) = I$ is called the *transition matrix* associated with this system. This solution is denoted by $\Phi_A(t, t_0)$ and exists when $A \in \mathcal{R}$. A matrix is regressive if and only if all of its eigenvalues are in $\mathcal{R}$. Equivalently, the matrix $A(t)$ is regressive if and only if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$.

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